CORE
Metadata, citation and similar papers at core.ac.uk
Provided by CONICET Digital

BMO spaces related to Laguerre semigroups.

Anibal Chicco Ruiz and Eleonor Harboure

Abstract

For the system of Laguerre functions $\{\varphi_n^\alpha\}$ we define a suitable BMO space from the atomic version of the Hardy space $H^1_{\varphi^\alpha} = \{f \in L^1 : \mathrm{W}^*_{\varphi^\alpha} f \in L^1\}$ considered by Dziubański in [7], where $\mathrm{W}^*_{\varphi^\alpha}$ is the maximal operator of the Heat Semigroup associated to that Laguerre system. We prove boundedness of $\mathrm{W}^*_{\varphi^\alpha}$ over a weighted version of that BMO, and we extend such result to other systems of Laguerre functions, namely $\{\mathcal{L}_n^\alpha\}$ and $\{\ell_n^\alpha\}$. To do that, we work with a more general family of weighted BMO-like spaces that includes those associated to all of the above mentioned Laguerre systems. In this setting, we prove that the local versions of the Hardy-Littlewood and the Heat-diffusion maximal operators turn to be bounded over such family of spaces for A^1_{loc} weights. This result plays a decisive role in proving the boundedness of Laguerre semigroup maximal operators.

1 Introduction.

For $\alpha > -1$, let us consider the Laguerre semigroups generated by the second order differential operators

$$L_\varphi = \frac{1}{4}\left\{-\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\alpha^2 - \frac{1}{4}\right)\right\}, \quad x > 0,$$

$$L_{\mathcal{L}} = -x\frac{d^2}{dx^2} - \frac{d}{dx} + \frac{x}{4} + \frac{\alpha^2}{4x}, \quad x > 0$$

and

$$L_\ell = -x\frac{d^2}{dx^2} - (\alpha+1)\frac{d}{dx} + \frac{x}{4}, \quad x > 0.$$

As it is well known, the eigenfunctions of these operators are given by

$$\varphi_n^\alpha(x) = \left(\frac{2n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x^2)e^{-x^2/2}x^{\alpha+1/2}, \tag{1.1}$$

$$\mathcal{L}_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x)e^{-x/2}x^{\alpha/2} \tag{1.2}$$

and

$$\ell_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x)e^{-x/2}, \tag{1.3}$$

respectively, where $L_n^\alpha(x)$ is the Laguerre polynomial of order n. The eigenvalues are, in the three cases, $n + \frac{\alpha+1}{2}$, for $n = 0, 1, 2, \ldots$. All systems give orthonormal basis of $L^2(\mathbb{R}^+)$, with the Lebesgue measure in the first two cases and with $x^\alpha dx$ in the last one.

Let us remember that, whenever we have $\{\psi_n\}$ an orthonormal basis of $L^2(d\mu)$, which members are eigenfunctions of a self-adjoint and non-negative second order differential operator L, with eigenvalues $\{\lambda_n\}$, we can define the **Heat-Diffusion Semigroup** $\{e^{-tL}\}_{t>0}$ as

$$e^{-tL}f(x) = \sum_n e^{-t\lambda_n} \langle f, \psi_n\rangle \, \psi_n(x)$$

and the **Maximal Operator** associate to this semigroup as

$$\mathrm{W}^* f(x) = \sup_{t>0} |e^{-tL} f(x)|.$$

In [14], [10], [11], [12] and [3], among others, the behaviour on Lebesgue and weighted Lebesgue spaces of the maximal semigroup operators associated to the above Laguerre systems $\{\varphi_n^\alpha\}$, $\{\mathcal{L}_n^\alpha\}$, and $\{\ell_n^\alpha\}$ denoted $\mathrm{W}^*_{\varphi^\alpha}$, $\mathrm{W}^*_{\mathcal{L}^\alpha}$ and $W^*_{\ell^\alpha}$ respectively, has been studied.

As it was pointed out in [14], all of the three semigroups are given by integration against explicit kernels. These kernels, near the diagonal, more precisely on the set $\Delta_2 = \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \frac{x}{2} < y < 2x\}$, behave very much like the classical Weierstrass kernel, and therefore the local parts of the maximal operators end up to be bounded by the Hardy-Littlewood maximal function but localized according to that region. More generally, as it was defined in [13], for any $\kappa > 1$, the **κ-Local Hardy-Littlewood Maximal operator** is given by

$$\mathrm{M}^\kappa_{loc} f(x) = \sup_{x\in I \in \mathcal{I}_\kappa} \frac{1}{|I|} \int_I |f(y)| dy, \tag{1.4}$$

for any $f \in L^1_{loc}(\mathbb{R}^+)$ and $x \in \mathbb{R}^+$, where

$$\mathcal{I}_\kappa = \{(a,b) : 0 < a < b \le \kappa a\} \tag{1.5}$$

is the set of **κ-local intervals on** $\mathbb{R}^+$.

The aim of this paper is to study the behaviour of the above maximal operators, $\mathrm{W}^*_{\varphi^\alpha}$, $\mathrm{W}^*_{\mathcal{L}^\alpha}$ and $W^*_{\ell^\alpha}$, acting on appropriate versions of weighted BMO spaces. Such spaces are naturally defined as duals of the H_1 spaces introduced by Dziubański in [7].

In fact we introduce a wider class of weighted BMO type spaces, in the spirit of those BMO considered in [8], that includes those associated to the Laguerre semigroups, and prove some special properties in section 2. Then, in this general context, we obtain in section 3 the boundedness over those weighted BMO-like spaces of the local Hardy-Littlewood maximal function M^κ_{loc} and of the **Local Maximal Heat-Diffusion Semigroup** T^*_{loc}, given by

$$\mathrm{T}^*_{loc} f(x) = \sup_{0<s<1} \left| \int_{\frac{x}{2}}^{2x} T_s(x,y) f(y) dy \right|, \tag{1.6}$$

where $T_s(x,y)$ is the classical heat-diffusion kernel

$$T_s(x,y) = \frac{1}{\sqrt{4\pi s}} \, e^{-\frac{|x-y|^2}{4s}}, \tag{1.7}$$

provided that the weight ω satisfies the A_1-Muckenhoupt condition only over local intervals, that is, those (a,b) with $0<a<b<2a$. We believe that these boundedness results may be of independent interest.

In sections 4, 5 and 6, we consider the particular cases of weighted BMO spaces associated to the three Laguerre systems. We are able to establish the boundedness of the maximal operators of the semigroups of each system over the corresponding spaces and under appropriate assumptions on the weight. The result concerning the continuity of the local maximal heat semigroup T^*_{loc}, obtained in section 3, turns out to be crucial. The classes of weights have a look resembling Muckenhoupt classes but weights in there may increase as any power at infinity. As an example, weights of the type $1+x^\gamma$ for any $\gamma\in\mathbb{R}$ are allowed in our classes.

Finally, as it was pointing out by one of the referees, two related articles by L.Cha and H.Liu have been published during the reviewing process of our manuscript. Both concern with BMO spaces associated to the Laguerre systems $\{\varphi_n^\alpha\}$, $\alpha>-1/2$. In [6] the authors prove the boundedness of the corresponding maximal semigroup operator on a BMO-like space, previously introduced in [5]. In fact, such space coincides with the one presented here in Section 4 for the case $\omega\equiv 1$ and hence our result stated as Theorem 4.3 is in fact a weighted version of Theorem 3.2 of [6]. However, let us remark that their technique is different from ours in the sense that they compare this Laguerre semigroup with the one dimensional Hermite semigroup. Such relationship was discovered in [1], where some clue estimates are obtained. Instead, our argument is based on local comparison with the classical heat semigroup and to do that we prove all the needed estimates. Let us add that also in Theorem 1 of [5], the authors actually prove that BMO_{L^α}, as they denote it, is the dual of the Hardy space $H^1_{L^\alpha}$ introduced in [7]. One ward of alert about their notation: even in [5] and in [4] the authors name the space as BMO_{L^α}, there is not an actual dependence from the parameter α, as the notation may suggests. Also we notice that from the atomic decomposition given in [7], the corresponding Hardy spaces are all the same up to a Banach spaces isomorphism.

2 $BMO_\tau(\omega)$ spaces.

In this section we will introduce the notion of local A^p classes of weights as well as the critical radius function and weighted versions of the BMO spaces associated to such function. Also, we will establish some basic but useful properties for them.

We start by reminding the definition of the classical $BMO(\mathbb{R}^+)$ space and its weighted version. Given a weight ω, we say that a locally integrable function on $\mathbb{R}^+=(0,\infty)$ belongs to $BMO(\omega)$ if it satisfies the bounded mean oscillation condition: there exists a constant C such that

$$\frac{1}{\omega(I)}\int_I |f(x)-f_I|dx\le C, \tag{2.1}$$

for all intervals I with closure contained in $\mathbb{R}^+$, where, as usual, $f_I=\frac{1}{|I|}\int_I f(x)dx$, that is the mean value of f over I. The seminorm $\|f\|_{BMO(\omega)}$ is taken as the least value of C that satisfies (2.1). In order to obtain a normed space, those functions which differ a.e. by a constant should be considered identical.

For the heat semigroup, the kind of weights that allow to extend important properties of the operators to weighted-BMO, are those in the Muckenhoupt classes. Let us remind the definition of **Muckenhoupt classes** A_1 and A_p, $1 < p < \infty$.

- A weight ω belongs to $A^p(\mathbb{R}^+)$, $1 < p < \infty$, if there exists $C > 0$ such that

$$\left(\int_I \omega(x)dx\right)^{1/p} \left(\int_I \omega(x)^{-p'/p}dx\right)^{1/p'} \leq C|I| \tag{2.2}$$

  for any interval $I \subset\subset \mathbb{R}^+$.

- A weight ω belongs to $A^1(\mathbb{R}^+)$, if there exists $C > 0$ such that

$$\omega(I) \leq C|I| \inf_{x\in I} \omega(x) \tag{2.3}$$

  for any interval $I \subset\subset \mathbb{R}^+$. By *inf* we mean the *essential infimum.*

- We denote $A^\infty = \bigcup_{p\geq 1} A^p$.

Our new kind of BMO type spaces will be defined for a wider classes of weights, namely the local Muckenhoupt classes as the ones considered in [13], section 6.

To be precise, given $\kappa > 1$, a weight ω on $\mathbb{R}^+$, i.e. any non-negative and $\mathbb{R}^+$-locally integrable function, is said to belong to $A^p_{loc,\kappa}$, $1 < p < \infty$, if there exists a constant $C = C(\kappa, p)$ such that (2.2) holds for any $B \in \mathcal{I}_\kappa$, being $\mathcal{I}_\kappa$ the set of κ-local intervals given by (1.5).

Similarly, for $p = 1$, we say that $\omega \in A^1_{loc,\kappa}$ if (2.3) holds for all $B \in \mathcal{I}_\kappa$.

The semi-norm $[\omega]_{p,\kappa}$ is the least constant $C(\kappa, p)$ for which (2.2) or (2.3) holds, according to $p > 1$ or $p = 1$. As usual, we denote $A^\infty_{loc,\kappa} = \bigcup_{p\geq 1} A^p_{loc,\kappa}$. From Proposition 6.1 of [13], the class $A^p_{loc,\kappa}$ actually does not depend on κ, and then it will be denoted just by A^p_{loc} and we shall say that ω is a **local** A_p **weight** whenever $\omega \in A^p_{loc}$. Nevertheless, the semi-norms $[\omega]_{p,\kappa}$ still depend on κ and may increase to infinity. This is the case when $\omega(x) = \frac{1}{x}$: it is not difficult to show that $\omega \in A^2_{loc,\kappa}$, for any κ, and $[\omega]_{2,\kappa} \to \infty$ when $\kappa \to \infty$.

In the same article, the authors established a relationship between those weights and the Local Maximal Hardy-Littlewood operator M^κ_{loc} given by (1.4). Indeed, they proved that M^κ_{loc} is of strong type (p, p), when $1 < p < \infty$, and of weak type $(1, 1)$, with respect to measure $\omega(x)dx$, if and only if $\omega \in A^p_{loc}$ or $\omega \in A^1_{loc}$, respectively.

Let us point out that if $\omega \in A^1_{loc}$, then it follows directly from definition that

$$\omega(I) \leq C_\kappa \frac{|I|}{|S|}\omega(S) \tag{2.4}$$

for any $I \in \mathcal{I}_\kappa$ and any measurable set $S \subset I$.

Moreover, as it was shown in [13], this property also holds for any $1 < p < \infty$. We shall refer to that as the **local doubling property**.

Lemma 2.1. *Let $\omega \in A^p_{loc}$, $1 \leq p < \infty$. Then, for every $\kappa > 1$, there exists a constant C_κ depending on κ, p and $[\omega]_{p,\kappa}$, such that*

$$\omega(I) \leq C_\kappa \left(\frac{|I|}{|S|} \right)^p \omega(S),$$

for any $I \in \mathcal{I}_\kappa$ and any measurable set $S \subset I$.

We introduce now the notion of critical radius function, that will be needed in the definition of our BMO spaces.

Definition 1. Given a positive and continuous function τ defined on $\mathbb{R}^+ = (0, \infty)$, we say that τ is a **critical radius function** if

$$\lim_{x \to 0^+} \tau(x) = 0, \tag{2.5}$$

and

$$\tau(y) \leq \tau(x) + \gamma |x - y|, \tag{2.6}$$

for some $0 < \gamma < 1$ and any $x, y \in \mathbb{R}^+$.

Examples of critical radius functions are, for $\gamma < 1$, $\tau(x) = \gamma x$, $\tau(x) = \gamma \min\{x, 1\}$ and $\tau(x) = \gamma \min\{x, \frac{1}{x}\}$.

Associated to a critical radius function we distinguish different types of intervals. Let us remark that we will always consider intervals $I = B(x, R) = (x - R, x + R)$ such that $\overline{I} \subset \mathbb{R}^+$, so we assume $0 < R < x$.

- *Critical interval*: $I = B(x, \tau(x)) = (x - \tau(x), x + \tau(x))$;
- *Sub-critical interval*: $I = B(x, R)$ such that $0 < R < \tau(x)$;
- *Super-critical interval*: $I = B(x, R)$ such that $R > \tau(x)$;
- λ*-super-critical interval*: $I = B(x, R)$ such that $R > \lambda \tau(x)$, where $0 < \lambda \leq 1$ is a fixed constant. In other words, I is a λ-super-critical interval if and only if I is super-critical for $\tau' = \lambda \tau$.

We enumerate some useful properties of τ and the related intervals. Their proofs are quite straightforward so we omit them.

Proposition 2.2. *Let τ satisfy (2.5) and (2.6). Then*

a)
$$\tau(x) \leq \gamma x, \quad \textit{for all } x \in \mathbb{R}^+. \tag{2.7}$$

b) Let $\kappa \doteq \frac{1+\gamma}{1-\gamma}$. If I is a critical or sub-critical interval for τ, then I is a κ-local interval (see (1.5)). Moreover, if I and J are two critical or sub-critical intervals for τ such that $\overline{I} \cap \overline{J} \neq \emptyset$, then $I \cup J$ is a κ^2- local interval.

c) If I is a critical interval for τ, then $\frac{1}{\kappa}\tau(x) \leq \tau(y) \leq \kappa \tau(x)$ for any $x, y \in \overline{I}$, where $\kappa \doteq \frac{1+\gamma}{1-\gamma}$. Moreover, if I and J are two critical intervals for τ such that $\overline{I} \cap \overline{J} \neq \emptyset$, then $\frac{1}{\kappa^2}\tau(x) \leq \tau(y) \leq \kappa^2 \tau(x)$, for any $x, y \in \overline{I \cup J}$.

The following statement gives a covering of $\mathbb{R}^+$ by means of critical intervals. We provide an elementary and constructive proof of this fact.

Lemma 2.3. *There exists an increasing sequence* $\{a_j\}_{j\in\mathbb{Z}}$ *of positive numbers such that the critical intervals* $I_j = (a_j - \tau(a_j), a_j + \tau(a_j))$ *are disjoint and satisfy* $\bigcup_{j\in\mathbb{Z}} \overline{I_j} = \mathbb{R}^+$.

Proof. In order to define the sequence $\{a_j\}_{j\in\mathbb{Z}}$, we first consider $j = 0$ and set $a_0 = 1$ and $I_0 = (1 - \tau(1), 1 + \tau(1))$. Since $\tau(x) \leq \gamma x$, for some fixed $0 < \gamma < 1$, we have $1 - \tau(1) \geq 1 - \gamma > 0$ and this implies $I_0 \subset\subset \mathbb{R}^+$.

For $j \geq 0$, we define a_{j+1} in order to satisfy $a_{j+1} > a_j$ and

$$a_j + \tau(a_j) = a_{j+1} - \tau(a_{j+1}). \tag{2.8}$$

In this way, the interval I_{j+1} is at the right of I_j and they have an extreme point in common.

In order to choose such a_{j+1}, we call a function $h(x) \doteq x - \tau(x)$ and a constant $b \doteq a_j + \tau(a_j)$. Note that h is continuous and $\lim_{x\to\infty} h(x) = +\infty$, since $h(x) \geq (1-\gamma)x$, for any $x > 0$. Then, since $b > h(a_j)$, there exists at least one $y > a_j$ such that $h(y) = b$. If we take $a_{j+1} = \inf\{y > a_j : h(y) = b\}$, this a_{j+1} will satisfy (2.8).

Now, in a similar way, we define a_{j-1} for $j \leq 0$ such that $a_{j-1} < a_j$ and

$$a_{j-1} + \tau(a_{j-1}) = a_j - \tau(a_j). \tag{2.9}$$

For that we consider $h(x) = x + \tau(x)$ and $b = a_j - \tau(a_j)$. Then, since h is continuous, $\lim_{x\to 0^+} h(x) = 0$ and $0 < b < h(a_j)$, we can take $a_{j-1} = \sup\{0 < y < a_j : h(y) = b\}$.

Thus, we have obtained a sequence $\{a_j\}_{j\in\mathbb{Z}}$ satisfying (2.8) and (2.9).

Finally, in order to prove that $\{\overline{I_j}\}$ cover $\mathbb{R}^+$, is enough to show that

$$\lim_{j\to+\infty} a_j = +\infty \tag{2.10}$$

and

$$\lim_{j\to-\infty} a_j = 0. \tag{2.11}$$

Both limits exist since $\{a_j\}_{j\in\mathbb{Z}}$ is increasing and takes values on $\mathbb{R}^+$. Suppose that $\lim_{j\to+\infty} a_j = b < +\infty$. Then, taking $j \to +\infty$ on (2.8), we obtain $b - \tau(b) = b + \tau(b)$ and this implies $\tau(b) = 0$. This cannot happen for any $b > 0$ since τ is a radius function, defined to be positive in $\mathbb{R}^+$. Thus he have obtained (2.10).

Analogously, if we assume $\lim_{j\to-\infty} a_j = a > 0$, making $j \to -\infty$ on (2.9) we obtain $\tau(a) = 0$. Thus, (2.11) holds.

$\square$

In the next lemma we show how to measure with a local weight λ-super-critical intervals for τ , using the covering just given.

Lemma 2.4. *Let $\omega \in A^{\infty}_{loc}$, τ a critical radius function and I a λ-super-critical interval for τ. If $\mathcal{J} = \{j \in \mathbb{Z} : I_j \cap I \neq \emptyset\}$, where $\{I_j\}$ is the covering by critical intervals of Lemma 2.3, then*

$$\omega(I) \leq \sum_{j \in \mathcal{J}} \omega(I_j) \leq C\omega(I), \tag{2.12}$$

for some constant C depending on λ, the constant γ of (2.6), and $[\omega]_{p,\kappa}$, with p such that $\omega \in A^p_{loc}$ and $\kappa = \left(\frac{1+\gamma}{1-\gamma}\right)^2$.

Proof. Since for any interval $I \subset\subset \mathbb{R}^+$ we have $I = \bigcup_{j\in\mathcal{J}} I \cap \overline{I_j}$, the first inequality is trivial.

Let $I = B(x_0, R)$, with $x_0 \in \mathbb{R}^+$ and $\lambda\tau(x_0) \leq R < x_0$. Suppose first that $\sharp\mathcal{J} = 1$. In this case $I \subset I_j$, for some $j \in \mathbb{Z}$. Also we have $\omega \in A^p_{loc}$, for some $1 \leq p < \infty$. Then, since by Proposition 2.2 b) I_j is a $\frac{1+\gamma}{1-\gamma}$-local interval, Lemma 2.1 gives us

$$\omega(I_j) \leq C\left(\frac{\tau(a_j)}{R}\right)^p \omega(I).$$

Then, since $R \geq \lambda\tau(x_0)$ and, by Proposition 2.2 c), $\tau(x_0) \simeq \tau(a_j)$, we obtain the second inequality of (2.12).

Suppose now that $\sharp\mathcal{J} = 2$. Then $I \subset \overline{I_j \cup I_{j+1}}$, for some integer j. Since, by Proposition 2.2 b) and c), $\overline{I_j \cup I_{j+1}}$ is a $\left(\frac{1+\gamma}{1-\gamma}\right)^2$-local interval and $\tau(a_j) \simeq \tau(a_{j+1}) \simeq \tau(x_0)$, by Lemma 2.1 we obtain again (2.12).

Finally, suppose $\sharp\mathcal{J} > 2$. Let us call j_0 to the first integer of $\mathcal{J}$ and j_1 to the last one. If j is such that $j_0 < j < j_1$ then $I_j \subset I$ and since all the I_j are disjoint, we can always write

$$\sum_{j\in\mathcal{J}} \omega(I_j) \leq \omega(I_{j_0}) + \omega(I) + \omega(I_{j_1}). \tag{2.13}$$

On the other hand, using again Lemma 2.1 and Proposition 2.2 we have

$$\begin{aligned} \omega(I_{j_0}) &\leq \omega(I_{j_0} \cup I_{j_0+1}) \\ &\leq C\omega(I_{j_0+1}) \\ &\leq C\omega(I). \end{aligned}$$

Analogously, $\omega(I_{j_1}) \leq C\omega(I_{j_1-1}) \leq C\omega(I)$. Therefore, from (2.13) we obtain (2.12). □

Now we are ready to introduce the spaces $BMO_\tau(\omega)$. As we noticed, it will be in the spirit of the BMO spaces, associated to some critical radius function, introduced in [8].

Definition 2. Let τ be a critical radius function and ω a weight in $\mathbb{R}^+$. We say that a real function $f \in L^1_{loc}(\mathbb{R}^+)$ belongs to $BMO_\tau(\omega)$ if there exists a constant C such that f satisfies the *bounded mean oscillation condition*

$$\frac{1}{\omega(I)} \int_I |f(y) - f_I| dy \leq C, \tag{2.14}$$

for any subcritical interval I (see definitions under equation (2.6), and the *bounded mean condition*

$$\frac{1}{\omega(I)}\int_I |f(y)|dy \le C, \tag{2.15}$$

for any critical and super-critical interval I. The norm $\|f\|_{BMO_\tau(\omega)}$ is taken as the least constant C satisfying both conditions.

Remark 2.5. Since $\int_I |f(y)-f_I|dy \le 2\int_I |f(y)|dy$ for any interval I, we have $BMO_\tau(\omega) \subset BMO(\omega)$. Also, $L^\infty(\omega^{-1}) = \{f : f\omega^{-1} \in L^\infty\} \subset BMO_\tau(\omega)$, since $\frac{1}{\omega(I)}\int_I |f(x)|dx \le \|f\omega^{-1}\|_\infty$, for any interval $I \subset\subset \mathbb{R}^+$.

Remark 2.6. Notice that if we ask condition (2.15) to be true only for super-critical intervals, by continuity it will also hold for critical intervals.

Remark 2.7. In [4], we introduced a local BMO space on $\mathbb{R}^+$ called $BMO^\kappa_{loc}(\omega)$, for $\kappa > 1$, associated to the family of intervals $\mathcal{I}_\kappa$, as those functions satisfying the bounded mean oscillation condition for intervals belonging to $\mathcal{I}_\kappa$, and the bounded mean condition for bigger intervals. The relationship between $BMO_\tau(\omega)$ and $BMO^\kappa_{loc}(\omega)$ is as follows. Given $\kappa > 1$, if we take $\tau_0(x) = \gamma x$ with γ satisfying $\kappa = \frac{1+\gamma}{1-\gamma}$, then the set of all sub-critical and critical intervals for τ_0 is exactly $\mathcal{I}_\kappa$, the set of κ-local intervals given by (1.5). Hence $BMO_\tau(\omega)$ is the same space as $BMO^\kappa_{loc}(\omega)$. More generally, for any critical radius function τ satisfying (2.5) and (2.6), and for $\kappa \ge \frac{1+\gamma}{1-\gamma}$, we have

$$BMO_\tau(\omega) \subset BMO^\kappa_{loc}(\omega), \tag{2.16}$$

in view of (2.7), $\gamma \le \frac{\kappa-1}{\kappa+1}$ and the obvious fact that $\tau \le \tau'$ implies $BMO_\tau \subset BMO_{\tau'}$.

The introduction of these spaces is inspired, as we said, by the study of the right substitutes of $BMO(\omega)$ for the context of the semigroups associated to the Laguerre systems $\{\varphi_n^\alpha\}$, $\{\mathcal{L}_n^\alpha\}$ and $\{\ell_n^\alpha\}$. Indeed, if we take $\rho(x) = \frac{1}{8}\min\{x, \frac{1}{x}\}$ and $\omega \equiv 1$, BMO_ρ is the dual of the atomic space $H^1_{L^\alpha}$ associated to $\{\varphi_n^\alpha\}$, studied by J. Dziubański in [7], for $\alpha > -\frac{1}{2}$. Also, for $\sigma(x) = \frac{1}{8}\min\{x, 1\}$, we obtain the proper BMO-spaces for the two other systems $\{\mathcal{L}_n^\alpha\}$. Later we will go over those particular cases and we shall study the action of the corresponding semigroup maximal operators on such spaces.

Now we establish some useful properties of $BMO_\tau(\omega)$.

The following lemma says that it is enough to check the bounded mean condition (2.15) just for critical intervals to conclude that it also holds for any supercritical interval.

Lemma 2.8. *Let $\omega \in A^\infty_{loc}$ and τ a critical radius function. Suppose that f, a locally integrable function on $\mathbb{R}^+$, satisfies*

$$\frac{1}{\omega(I)}\int_I |f(x)|dx \le A \tag{2.17}$$

for all critical intervals $I \subset\subset \mathbb{R}^+$, where A is a constant depending on f and ω. Then, for each $0 < \lambda < 1$, (2.17) also holds for any λ-super-critical interval, with constant CA, where C is the constant of Lemma 2.4.

Proof. Let I a λ-supercritical interval and let $\mathcal{J} = \{j \in \mathbb{Z} : I_j \cap I \neq \emptyset\}$. Since each I_j is a critical interval, by hypothesis we obtain

$$\begin{aligned} \int_I |f(x)|dx &\leq \sum_{j\in\mathcal{J}} \int_{I_j} |f(x)|dx \\ &\leq A \sum_{j\in\mathcal{J}} \omega(I_j) \\ &\leq AC\omega(I), \end{aligned}$$

where the last inequality arises by Lemma 2.4. □

As immediate consequences we obtain:

Corollary 2.9. *Let $\omega \in A^{\infty}_{loc}$ and $f \in L^1_{loc}(\omega)$ such that (2.14) holds for any sub-critical interval respect some critical radius function τ. Then, $f \in BMO_\tau(\omega)$ if and only if f satisfies the bounded mean condition (2.15) for any critical interval for τ.*

Corollary 2.10. *If $\omega \in A^{\infty}_{loc}$ and $f \in BMO_\tau(\omega)$ then*

$$\frac{1}{\omega(I)} \int_I |f(x)|dx \leq C_\lambda \|f\|_{BMO_\tau(\omega)},$$

for any λ-supercritical interval I.

We usually say that two non-negative functions f and g are equivalent, denoted $f \simeq g$, if there exist constants c and C such that $cf(x) \leq g(x) \leq Cf(x)$ for *a.e.* x for which f and g are defined.

Corollary 2.11. *If $\tau \simeq \tau'$ and $\omega \in A^{\infty}_{loc}$ then $BMO_\tau(\omega) = BMO_{\tau'}(\omega)$, with equivalence of norms depending on the constants of the relation between τ and τ'. In particular, all the spaces $BMO^{\kappa}_{loc} = BMO^{\kappa'}_{loc}$ contain the same functions, for any $\kappa > 1$.*

Proof. Let $\tau \lesssim \tau'$ and $f \in BMO_\tau(\omega)$. In order to obtain $f \in BMO_{\tau'}(\omega)$, by Corollary 2.9, we only have to prove that (2.15) holds for $I = B(x_0, \tau'(x_0))$, with $x_0 \in \mathbb{R}^+$. Since $\tau'(x_0) \geq c\tau(x_0)$, I is a c-super-critical interval for τ and the result follows from Corollary 2.10. □

Remark 2.12. Notice that given τ and $0 < \lambda < 1$, λ-supercritical intervals become supercritical with respect to $\tau_\lambda(x) \doteq \lambda\tau(x)$. By Corollary 2.11, we have $BMO_{\tau_\lambda} = BMO_\tau$. But we can not move λ too many times since the BMO-norm with respect to τ_λ may go to infinity. We already remark that a similar thing happens with local weights: although $A^p_{loc,\kappa}$ contains the same functions for any $\kappa > 1$, we can find a weight such that the $A^p_{loc,\kappa}$-norm increase to infinity with κ (just consider $\omega(x) = \frac{1}{x}$). For that reason, many times we will work out our proofs with the explicit values of κ and λ that we need to consider in order to get the desired results.

The following lemma extends the familiar consequence of John-Nirenberg inequality for classic BMO to the space $BMO_\tau(\omega)$.

Lemma 2.13 (Equivalence of norm's property.). *Let $\omega \in A^p_{loc}$, $1 < p < \infty$, and τ a critical radius function. For $1 \leq r \leq p'$, there exists a constant $C = C(r, \omega, \tau)$ such that if $f \in BMO_\tau(\omega)$ then*

$$\left(\frac{1}{\omega(B)} \int_B |f(x) - f_B|^r \omega^{1-r}(x) dx \right)^{1/r} \leq C \|f\|_{BMO_\tau(\omega)} \tag{2.18}$$

for all critical and sub-critical intervals B, ie: $B = B(x_0, R)$ with $0 < R \leq \tau(x_0)$, and

$$\left(\frac{1}{\omega(B)} \int_B |f(x)|^r \omega^{1-r}(x) dx \right)^{1/r} \leq C \|f\|_{BMO_\tau(\omega)} \tag{2.19}$$

for all critical and super-critical intervals, ie: $B = B(x_0, R)$, with $R \geq \tau(x_0)$.

Proof. Let $f \in BMO_\tau(\omega)$. Then, by (2.16), $f \in BMO^\kappa_{loc}(\omega)$, with $\kappa = \frac{1+\gamma}{1-\gamma}$. In [4] we have proved, given $\kappa > 1$ and a weight $\omega \in A^p_{loc}$, for any r such that $1 \leq r \leq p'$, that

$$\left(\frac{1}{\omega(B)} \int_B |f(x) - f_B|^r \omega^{1-r}(x) dx \right)^{1/r} \leq C \|f\|_{BMO^\kappa_{loc}(\omega)},$$

for any κ-local interval B. Then, Proposition 2.2 b) imply (2.18) for any critical and sub-critical interval for τ.

We will prove now (2.19). Consider first $B = B(x_0, \tau(x_0))$. Then

$$\begin{aligned} \left(\frac{1}{\omega(B)} \int_B |f(x)|^r \omega^{1-r}(x) dx \right)^{1/r} &\leq \left(\frac{1}{\omega(B)} \int_B |f(x) - f_B|^r \omega^{1-r}(x) dx \right)^{1/r} \\ &+ \left(\frac{\omega^{1-r}(B)}{\omega(B)} \right)^{1/r} |f_B|. \end{aligned}$$

From (2.18), the first term on the right hand side is bounded by $\|f\|_{BMO_\tau(\omega)}$. For the second term, observe that $r \leq p'$ and $\omega \in A^p_{loc}$ imply $\omega^{1-r} \in A^r_{loc}$, and then

$$\begin{aligned} \left(\frac{\omega^{1-r}(B)}{\omega(B)} \right)^{1/r} |f_B| &\leq C \frac{1}{\omega(B)} \int_B |f(x)| dx \\ &\leq \|f\|_{BMO_\tau(\omega)}. \end{aligned}$$

Finally, if we consider $B = B(x_0, R)$ with $R > \tau(x_0)$, we use the result for critical intervals just proved to obtain

$$\begin{aligned} \int_B |f(x)|^r \omega^{1-r}(x) dx &\leq \sum_{j \in \mathcal{J}} \int_{I_j} |f(x)|^r \omega^{1-r}(x) dx \\ &\leq C \|f\|^r_{BMO_\tau(\omega)} \sum_{j \in \mathcal{J}} \omega(I_j), \end{aligned}$$

where $\mathcal{J} = \{j \in \mathbb{Z} : I_j \cap B(x_0, R) \neq \emptyset\}$ and $\{I_j\}$ is the covering by critical intervals. Then, using Lemma 2.4, we obtain (2.19). □

Finally, we state a version of a very well known and useful property for functions in $BMO(\omega)$ with $\omega \in A_1$. Because of our assumption $\omega \in A^1_{loc}$, we have to restrict the conclusion to local intervals. Its proof follows exactly with the same steps, so we omit it.

Lemma 2.14. *Consider two*
kappa-
local intervals J and J' with the same center such that $J \subset J'$. Then, if $f \in BMO_\tau(\omega)$ and $\omega \in A^1_{loc}$, we have

$$\int_{J'} |f(x) - f_J|dx \leq C_\kappa \|f\|_{BMO_\tau(\omega)} \omega(J) \frac{|J'|}{|J|} \ln\left(\frac{|J'|}{|J|} + 1\right). \tag{2.20}$$

3 Local Classical Operators on $BMO_\tau(\omega)$

In this section we will introduce the local versions of the Hardy-Littlewood Maximal function and the Heat Diffusion Maximal operator. We will establish their boundedness over $BMO_\tau(\omega)$ spaces.

Let us remind that in the classic BMO theory, the Hardy-Littlewood Maximal function M is not bounded on BMO, since we may have $\mathrm{M}f \equiv \infty$ for some $f \in BMO$ (see [2]). Anyway, from [4], it is already known that for $\omega \in A^1_{loc}$, the Local Maximal Hardy-Littlewood operator M^κ_{loc}, with $\kappa > 1$, given by (1.4), is bounded from $BMO^\kappa_{loc}(\omega)$ into $BMO^\kappa_{loc}(\omega)$, with boundedness constant depending on κ. As we already pointed out, such spaces coincide for different values of κ (corollary 2.11) and also are particular cases of our family BMO_τ, in fact they contain all of them (remark 2.7). Based on that, we will prove now a more general result.

Theorem 3.1. *Let $\kappa > 1$ and τ a critical radius function satisfying* (2.5) *and* (2.6)*, for some $0 < \gamma < 1$. Then, if $\omega \in A^1_{loc}$, the operator M^κ_{loc} is bounded on $BMO_\tau(\omega)$, with constant depending on κ, γ and the $A^1_{loc,\kappa}$ constant of ω.*

Proof. Fix $\kappa > 1$. Along the proof we assume that $\gamma \in (0,1)$ is such that $\kappa > \frac{1+\gamma}{1-\gamma}$. Then, by Corollary 2.11, we can extend the results for any $0 < \widetilde{\gamma} < 1$, considering $\widetilde{\tau} = \frac{\widetilde{\gamma}}{\gamma}\tau$.

Let $f \in BMO_\tau(\omega)$. By (2.16) we have $BMO_\tau(\omega) \subset BMO^\kappa_{loc}(\omega)$ continuously. This, together with the boundedness of M^κ_{loc} obtained in [4] gives that $\|\mathrm{M}^\kappa_{loc} f\|_{BMO^\kappa_{loc}(\omega)} \lesssim \|f\|_{BMO_\tau(\omega)}$, provided $w \in A^1_{loc}$. In particular, this implies that M^κ_{loc} is locally integrable and

$$\frac{1}{\omega(I)} \int_I |\mathrm{M}^\kappa_{loc} f(x) - (\mathrm{M}^\kappa_{loc} f)_I| dx \leq C\|f\|_{BMO_\tau(\omega)}$$

for any interval I compactly contained in $\mathbb{R}^+$.

On the other hand, by Corollary 2.9, it is enough to prove the bounded mean condition

$$\frac{1}{\omega(B)} \int_B |\mathrm{M}^\kappa_{loc} f(x)| dx \leq C\|f\|_{BMO_\tau(\omega)}, \tag{3.1}$$

for any critical interval $B = B(x_0, \tau(x_0))$. Let $B^* = B(x_0, \sigma\tau(x_0))$, where $\sigma = \frac{1}{\sqrt{\gamma}} > 1$. Let us write $f = f_1 + f_2$, where $f_1 = f\chi_{B^*}$ and $f_2 = f\chi_{B^{*c}}$.

For f_1, we apply Hölder inequality and we use that $\omega \in A^1_{loc}$ implies $\omega^{1-p} \in A^p_{loc}$ for any $p > 1$, and hence $\mathrm{M}^\kappa_{loc} : L^p(\omega^{1-p}) \longrightarrow L^p(\omega^{1-p})$. Therefore

$$\begin{aligned}
\frac{1}{\omega(B)} \int_B |\mathrm{M}^\kappa_{loc} f_1(x)| dx &\leq \left(\frac{1}{\omega(B)} \int |\mathrm{M}^\kappa_{loc} f_1(x)|^p \omega^{1-p}(x) dx \right)^{\frac{1}{p}} \\
&\leq C_\kappa \left(\frac{1}{\omega(B)} \int_{B^*} |f(x)|^p \omega^{1-p}(x) dx \right)^{\frac{1}{p}} \\
&\leq C \left(\frac{\omega(B^*)}{\omega(B)} \right)^{\frac{1}{p'}} \|f\|_{BMO_\tau(\omega)}.
\end{aligned}$$

Since $B^* \subset B(x_0, \sqrt{\gamma} x_0)$, Lemma 2.1 gives $\omega(B^*) \leq C_\gamma \omega(B)$. Then, from the equivalence of norm's inequality (2.19), (3.1) becomes true for f_1.

As for f_2 we will prove first that for $x \in B$

$$\mathrm{M}^\kappa_{loc} f_2(x) \leq C \, \|f\|_{BMO_\tau(\omega)} \sup_J \frac{\omega(J)}{|J|}, \tag{3.2}$$

where the supremum is taken over those $J \in \mathcal{I}_\kappa$ such that $x \in J$ and $J \cap B^{*c} \neq \emptyset$. Indeed, observe that to evaluate the left hand side for some $x \in B$ we only have to consider κ-local intervals J such that $J \cap B \neq \emptyset$ and $J \cap B^{*c} \neq \emptyset$. In this case we have

$$|J| > (\sigma - 1)\tau(x_0). \tag{3.3}$$

If x_J denotes the center of J, using (2.6) and (3.3) we obtain

$$\tau(x_J) \leq \gamma |x_J - x_0| + \tau(x_0) \leq C_\gamma |J|.$$

Then by Corollary 2.10, (3.2) follows.

Now, for each of those J of the supremum of (3.2), the interval $J' = J \cup B$, by Proposition 2.2 b), is a κ^2-local interval. Then by (2.4), and since $|J| \simeq |J'|$, we have $\frac{\omega(J)}{|J|} \lesssim \frac{\omega(J')}{|J'|} \lesssim \frac{\omega(B)}{|B|}$. This implies

$$\mathrm{M}^\kappa_{loc} f_2(x) \lesssim \|f\|_{BMO_\tau(\omega)} \frac{\omega(B)}{|B|},$$

and then (3.1) holds for f_2. □

Next we consider the local version of the classical heat-diffusion semigroup, and its associated maximal operator, T^*_{loc}, given by (1.6). As expected, it turns out that T^*_{loc} is controlled by some local Maximal Function. Such estimate together with Theorem 3.1 will help us to prove the boundedness of T^*_{loc} over $BMO_\tau(\omega)$.

Lemma 3.2. *There exists a constant C such that $\mathrm{T}^*_{loc} f(x) \leq C\mathrm{M}^4_{loc} f(x)$, for all $x \in \mathbb{R}^+$ and any f locally integrable function.*

Proof. Let f a locally integrable function. We have to check that

$$\int_{\frac{x}{2}}^{2x} T_s(x,y) |f(y)| dx \leq C\mathrm{M}^4_{loc} f(x),$$

for any $x \in \mathbb{R}^+$ and $s \in (0,1)$. For fixed x and s pick the integer j_0 such that $2^{j_0+1}\sqrt{s} \le x < 2^{j_0+2}\sqrt{s}$. Let us call $B_j = B(x, 2^j\sqrt{s})$. By our choice of j_0, we have that $\frac{x}{2} \le x - 2^{j_0}\sqrt{s} < \frac{3}{4}x$ and $\frac{5}{4}x < x + 2^{j_0}\sqrt{s} < 2x$. This choice gives us $(\frac{x}{2}, 2x) \subset (\frac{x}{2}, \frac{3}{4}x) \cup B_{j_0} \cup (\frac{5}{4}x, 2x)$ and we may write

$$\begin{aligned}\int_{\frac{x}{2}}^{2x} T_s(x,y)|f(y)|dy &\le \int_{\frac{x}{2}}^{\frac{3}{4}x} T_s(x,y)|f(y)|dy + \int_{B_{j_0}} T_s(x,y)|f(y)|dy \\ &\quad + \int_{\frac{5}{4}x}^{2x} T_s(x,y)|f(y)|dy \\ &= I(x) + II(x) + III(x).\end{aligned}$$

For $I(x)$ and $III(x)$ the estimate follows easily since in any case $T_s(x,y) \lesssim \frac{1}{x}$ and the intervals of integration are 4-local, contain the point x and their measures are equivalent to x.

On the other hand, we write

$$II(x) = \sum_{j=-\infty}^{j_0} \int_{B_j \setminus B_{j-1}} T_s(x,y)|f(y)|dy.$$

If $y \in B_j \setminus B_{j-1}$, for $j \le j_0$, we have $|y - x| \ge 2^{j-1}\sqrt{s}$ and this implies $T_s(x,y) \lesssim \frac{1}{\sqrt{s}} e^{-c2^{2j}}$. Then,

$$II(x) \lesssim \sum_{j=-\infty}^{j_0} e^{-c2^{2j}} 2^j \frac{1}{|B_j|} \int_{B_j} f(y)dy \lesssim M_{loc}^4 f(x),$$

since, for any $j \le j_0$, $B_j \subset (\frac{x}{2}, 2x)$ and hence they are 4-local intervals. □

Remark 3.3. Let us notice that $M_{loc}^4 f < \infty$ *a.e.* for any locally integrable function on $\mathbb{R}^+$. In fact, to evaluate $M_{loc}^4 f(x)$ for $x \in [2^j, 2^{j+1}], j \in \mathbb{Z}$, we may replace f by the integrable function $f\chi_{[2^{j-2}, 2^{j+3}]}$. Therefore the above lemma implies the same property for T_{loc}^*.

Now we will prove the most important result of this section.

Theorem 3.4. *If τ is a critical radius function and $\omega \in A_{loc}^1$, then T_{loc}^* is bounded on $BMO_\tau(\omega)$.*

Proof. Let $f \in BMO_\tau(\omega)$ and $B = B(x_0, R)$, with $x_0 \in \mathbb{R}^+$ and $R > 0$.

We will prove first that $\mathrm{T}_{loc}^* f$ satisfies the bounded mean condition (2.15) for $R \ge \frac{\tau(x_0)}{3}$. For this we use Lemma 3.2, which gives $\mathrm{T}_{loc}^* \lesssim \mathrm{M}_{loc}^\kappa$, and Theorem 3.1. Then, by Corollary 2.10, we obtain

$$\frac{1}{\omega(B)} \int_B |\mathrm{T}_{loc}^* f(x)| dx \lesssim \|f\|_{BMO_\tau(\omega)}$$

for any $\frac{1}{3}$-supercritical interval B, that is, with $R \ge \frac{\tau(x_0)}{3}$.

Since the bounded mean condition (2.15) implies the bounded oscillation condition (2.14), it only remains to prove that

$$\frac{1}{\omega(B)} \int_B |\mathrm{T}_{loc}^* f(x) - c| dx \le C\|f\|_{BMO_\tau(\omega)} \tag{3.4}$$

holds for $0 < R < \frac{\tau(x_0)}{3}$, and some $c = c(f, B)$.

By Corollary 2.11, we may assume $\gamma = \frac{1}{8}$ and hence $\tau(x) \leq \frac{x}{8}$, for any $x \in \mathbb{R}^+$.

Let us call $B^* = B(x_0, 3R)$ and $f = f_1 + f_2 + f_3$, where $f_1 = (f - f_{B^*})\chi_{B^*}$, $f_2 = (f - f_{B^*})\chi_{(B^*)^C}$ and $f_3 = f_{B^*}$, and let us choose $x_1 \in B(x_0, \frac{R}{3})$ such that $c \doteq \mathrm{T}^*_{loc}(f_2 + f_3)(x_1) < \infty$. Observe that, by the above Remark, $\mathrm{T}^*_{loc}(f_2 + f_3)$ is finite *a.e.* If we denote $\widetilde{T}_s f(x) \doteq \int_{x/2}^{2x} T_s(x, y) f(y) dy$, we have $\mathrm{T}^*_{loc} f(x) = \sup_{0<s<1} |\widetilde{T}_s f(x)|$. Then

$$|\mathrm{T}^*_{loc} f(x) - c| \leq A_1(x) + A_2(x) + A_3(x),$$

where

$$A_1(x) = \mathrm{T}^*_{loc} f_1(x),$$

$$A_2(x) = \sup_{0<s<1} |\widetilde{T}_s f_2(x) - \widetilde{T}_s f_2(x_1)|,$$

and

$$A_3(x) = \sup_{0<s<1} |\widetilde{T}_s f_3(x) - \widetilde{T}_s f_3(x_1)|.$$

In order to obtain (3.4), it is enough to prove for $i = 1, 2, 3$ that

$$\frac{1}{\omega(B)} \int_B A_i(x) dx \leq C \|f\|_{BMO_\tau(\omega)}. \tag{3.5}$$

For $A_1(x)$, observe that $\omega \in A^1_{loc}$ implies $\omega^{-1} \in A^2_{loc}$. Then, from [13], M^4_{loc} is of strong type $(2, 2)$ with weight ω^{-1}, and so is T^*_{loc}, according to Lemma 3.2. Then, using Hölder inequality, we have

$$\begin{aligned}
\frac{1}{\omega(B)} \int_B A_1(x) dx &\leq \left(\frac{1}{\omega(B)} \int |\mathrm{T}^*_{loc} f_1(x)|^2 \omega^{-1}(x) dx \right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{1}{\omega(B)} \int_{B^*} |f(x) - f_{B^*}|^2 \omega^{-1}(x) dx \right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{BMO_\tau(\omega)},
\end{aligned}$$

where in the last inequality we have used the local doubling property of ω (Lemma 2.1) and the equivalence of norms inequality (2.18).

Next we consider $A_3(x)$. First note that

$$|\widetilde{T}_s f_3(x) - \widetilde{T}_s f_3(x_1)| = |f_{B*}| \left| \int_{\frac{x}{2}}^{2x} T_s(x, y) dy - \int_{\frac{x_1}{2}}^{2x_1} T_s(x_1, y) dy \right|$$

Performing the changes of variables $z = \frac{y-x}{\sqrt{s}}$ and $z = \frac{y-x_1}{\sqrt{s}}$ in each integral, we obtain

$$\begin{aligned}
\left| \int_{\frac{x}{2}}^{2x} T_s(x, y) dy - \int_{\frac{x_1}{2}}^{2x_1} T_s(x_1, y) dy \right| &\lesssim \left| \int_{-\frac{x_1}{2\sqrt{s}}}^{-\frac{x}{2\sqrt{s}}} e^{-\frac{z^2}{4}} dz \right| + \left| \int_{\frac{x}{\sqrt{s}}}^{\frac{x_1}{\sqrt{s}}} e^{-\frac{z^2}{4}} dz \right| \\
&\lesssim \left| \int_{\frac{x}{\sqrt{s}}}^{\frac{x_1}{\sqrt{s}}} e^{-\frac{z^2}{16}} dz \right|.
\end{aligned}$$

Since x and x_1 belong to B, which is contained in $B(x_0, \frac{1}{3}\tau(x_0))$, B is local and then $x \simeq x_1 \simeq x_0$ which implies

$$\left| \int_{\frac{x}{\sqrt{s}}}^{\frac{x_1}{\sqrt{s}}} e^{-\frac{z^2}{16}} dz \right| \lesssim \frac{|x-x_1|}{\sqrt{s}} e^{-c\frac{x_0^2}{s}} \lesssim \frac{|B|}{x_0},$$

for some constant c. Then, since $B^* \subset B(x_0, \tau(x_0)) \subset (\frac{7}{8}x_0, \frac{9}{8}x_0) \doteq I_0$, which is a super-critical interval for τ, we have

$$\begin{aligned} |\widetilde{T}_s f_3(x) - \widetilde{T}_s f_3(x_1)| &\lesssim \frac{1}{x_0} \int_{I_0} |f(y)| dy \\ &\lesssim \|f\|_{BMO_\tau(\omega)} \frac{\omega(I_0)}{|I_0|} \\ &\lesssim \|f\|_{BMO_\tau(\omega)} \frac{\omega(B)}{|B|}, \end{aligned}$$

for any $x \in B$, where in the last inequality we use (2.4) and that I_0 is local. Therefore, (3.5) holds for A_3.

Finally, regarding A_2, we will show that, for any $x \in B$,

$$A_2(x) \leq \|f\|_{BMO_\tau(\omega)} \frac{\omega(B)}{|B|}, \tag{3.6}$$

which implies (3.5) for A_2.

Note that

$$\begin{aligned} A_2(x) &\leq \sup_{0<s<1} \int \left| T_s(x,y)\chi_{(\frac{x}{2},2x)} - T_s(x_1,y)\chi_{(\frac{x_1}{2},2x_1)} \right| |f_2(y)| dy \\ &\leq A_{21}(x) + A_{22}(x) + A_{23}(x), \end{aligned}$$

where

$$A_{21}(x) \doteq \sup_{0<s<1} \int_{(\frac{x}{2},2x)\setminus(\frac{x_1}{2},2x_1)} T_s(x,y)\ |f_2(y)| dy,$$

$$A_{22}(x) \doteq \sup_{0<s<1} \int_{(\frac{x_1}{2},2x_1)\setminus(\frac{x}{2},2x)} T_s(x_1,y)\ |f_2(y)| dy$$

and

$$A_{23}(x) \doteq \sup_{0<s<1} \int_{(\frac{x}{2},2x)\cap(\frac{x_1}{2},2x_1)} |T_s(x,y) - T_s(x_1,y)|\, |f_2(y)| dy.$$

Let us call $B_x = (\frac{x}{2}, 2x) \setminus (\frac{x_1}{2}, 2x_1)$, with $x \neq x_1$. Notice that $x, x_1 \in (\frac{7}{8}x_0, \frac{9}{8}x_0)$ imply $(\frac{x}{2}, 2x) \subset (\frac{7}{16}x_0, \frac{9}{4}x_0)$ and $(\frac{9}{16}x_0, \frac{7}{4}x_0) \subset (\frac{x_1}{2}, 2x_1)$. Then, for $y \in B_x$ we have $|x-y| \geq |y-x_0| - |x-x_0| \geq \frac{3}{4}x_0$ and hence $T_s(x,y) \leq C\frac{1}{x_0}$. Therefore

$$\begin{aligned} A_{21}(x) &\lesssim \frac{1}{x_0} \int_{B_x} |f(y) - f_{B^*}| dy \\ &\lesssim \frac{1}{x_0} \int_{B_x} |f(y)| dy + \frac{|B_x|}{x_0} |f_{B^*}|. \end{aligned}$$

Since $|B_x| \leq C|x-x_1| < C|B|$ and the local interval $J_0 = \left(\frac{7}{16}x_0, \frac{9}{4}x_0\right)$ contains B_x and B^* we have

$$\begin{aligned} A_{21}(x) &\leq C\frac{1}{x_0}\int_{J_0}|f(y)|dy \\ &\leq \|f\|_{BMO_\tau(\omega)}\frac{\omega(J_0)}{|J_0|}, \end{aligned}$$

and we obtain (3.6) for A_{21} using again (2.4). In an analogous way, we obtain the same for A_{22}.

We will prove now (3.6) for A_{23}. First notice that $B^* \subset (\frac{7}{8}x_0, \frac{9}{8}x_0)$ and $(\frac{9}{16}x_0, \frac{7}{4}x_0) \subset \left(\frac{x}{2}, 2x\right) \cap \left(\frac{x_1}{2}, 2x_1\right) \subset (\frac{7}{16}x_0, \frac{9}{4}x_0)$. This implies that $\left(\frac{x}{2}, 2x\right) \cap \left(\frac{x_1}{2}, 2x_1\right) \cap B^{*c}$ is not empty and is contained in $I_0 \setminus B^*$, where $I_0 = (\frac{7}{16}x_0, \frac{9}{4}x_0)$.

On the other hand, since T^* is a vector valued Calderón-Zygmund operator, or also applying the mean value Theorem to $T_s(x,y)$ in the variable x, we obtain

$$\sup_{0<s<1}|T_s(x,y)-T_s(x_1,y)| \leq C\frac{|x-x_1|}{|y-x_1|^2}$$

if $|y-x_1| > 2|x-x_1|$. This is actually the case for $x \in B$, $x_1 \in B(x_0, \frac{R}{3})$ and $y \in (B^*)^c$. Also, $|y-x_1| \simeq |y-x_0|$ and then

$$\sup_{0<s<1}|T_s(x,y)-T_s(x_1,y)| \leq C\frac{R}{|y-x_0|^2}.$$

Therefore, using that $\left(\frac{x}{2}, 2x\right) \cap \left(\frac{x_1}{2}, 2x_1\right) \subset I_0$, we have

$$A_{23}(x) \lesssim R\int_{I_0\setminus B^*}\frac{|f(y)-f_{B^*}|}{|y-x_0|^2}dy.$$

Let us call $B_j = B(x_0, 3^jR)$ and choose an integer j_0 such that $3^{j_0} < \frac{x_0}{8R} \leq 3^{j_0+1}$. Then

$$I_0 \subset \left(\frac{7}{16}x_0, \frac{7}{8}x_0\right)\bigcup B_{j_0+1}\bigcup\left(\frac{9}{8}x_0, \frac{9}{4}x_0\right).$$

Observe that, since $3R < \tau(x_0) \leq \frac{x_0}{8}$, we have $j_0 \geq 1$. Then we can write

$$A_{23}(x) \lesssim \frac{R}{{x_0}^2}\int_{I_0}|f(y)-f_{B^*}|dy + R\int_{B_{j_0+1}\setminus B_1}\frac{|f(y)-f_{B^*}|}{|y-x_0|^2}dy.$$

The first term can be estimated as we did with $A_{21}(x)$. For the second we have

$$\begin{aligned} R\int_{B_{j_0+1}\setminus B_1}\frac{|f(y)-f_{B^*}|}{|y-x_0|^2}dy &\lesssim R\sum_{j=1}^{j_0}\int_{B_{j+1}\setminus B_j}\frac{|f(y)-f_{B^*}|}{|y-x_0|^2}dy \\ &\lesssim \sum_{j=1}^{j_0}\frac{1}{3^{2j}R}\int_{B_{j+1}}|f(y)-f_{B^*}|dy. \end{aligned}$$

Let us note that each B_{j+1} is a local interval since $B_{j_0+1} \subset I_0$. Then, using Lemma 2.14 with $J = B^* = B_1$ and $J' = B_{j+1}$, we get

$$\begin{aligned}
\sum_{j=1}^{j_0} \frac{1}{3^{2j}R} \int_{B_{j+1}} |f(y) - f_{B^*}| dy &\lesssim \|f\|_{BMO_\tau(\omega)} \frac{\omega(B)}{R} \sum_{j=1}^{j_0} \frac{j}{3^j} \\
&\lesssim \|f\|_{BMO_\tau(\omega)} \frac{\omega(B)}{|B|}.
\end{aligned}$$

Thus, we have obtained (3.6) for $A_{23}(x)$ completing the proof of the Theorem.

□

If, for a given critical radius function τ, we consider, instead of T^*_{loc}, the smaller operator

$$\mathrm{T}^*_{loc,\tau} f(x) \doteq \sup_{\tau(x)^2 \le s < 1} \left| \int_{\frac{x}{2}}^{2x} T_s(x,y) f(y) dy \right|, \tag{3.7}$$

we obtain the following stronger result that will be useful in the next section.

Proposition 3.5. *The operator $\mathrm{T}^*_{loc,\tau}$ is bounded from $BMO_\tau(\omega)$ into $L^\infty(\omega^{-1})$, for $\omega \in A^1_{loc}$.*

Proof. Let $f \in BMO_\tau(\omega)$. Without loss of generality, by Corollary 2.11, we may consider $\gamma = \frac{1}{8}$. Let us fix $x \in \mathbb{R}^+$ and $0 < s < 1$ such that x is a Lebesgue point of ω and $s \ge \tau(x)^2$. Notice that, for such x, we have $\inf_{y \in I} \omega(y) \le \omega(x)$, for any interval I which contains x. Remember that inf represents the essential infimum. Now, choose the integer $k_0 \le 0$ such that $2^{k_0} \le \frac{\tau(x)}{\sqrt{s}} < 2^{k_0+1}$ and let us call $B_k \doteq B(x, 2^k\sqrt{s}) \cap (\frac{x}{2}, 2x)$, for $k \ge k_0$. Observe that the B_k are increasing intervals and, after a certain k_1, they are equal to $(\frac{x}{2}, 2x)$. Since $B_{k+1} \setminus B_k = \{y \in (\frac{x}{2}, 2x) : 2^k \le \frac{|y-x|}{\sqrt{s}} < 2^{k+1}\}$ for $k_0 \le k \le k_1 - 1$, we can write

$$\begin{aligned}
\int_{\frac{x}{2}}^{2x} e^{-\frac{|y-x|^2}{4s}} |f(y)| dy &\le \int_{B_{k_0}} |f(y)| dy + \sum_{k=k_0}^{k_1-1} \int_{B_{k+1}\setminus B_k} e^{-\frac{|y-x|^2}{4s}} |f(y)| dy \\
&\le \int_{B_{k_0}} |f(y)| dy + \sum_{k=k_0}^{k_1-1} e^{-c\, 2^{2k}} \int_{B_{k+1}} |f(y)| dy
\end{aligned}$$

We will show now that each B_k, for $k_0 \le k \le k_1$, is a $\frac{8}{17}$-super-critical interval for τ. In fact, denote by b_k the center of B_k and R_k to it radius. Since $B(x, \frac{1}{2}\tau(x))$ is contained in $B(x, 2^{k_0}\sqrt{s})$ and also in $(\frac{x}{2}, 2x)$, for any $k \ge k_0$ we have $B(x, \frac{1}{2}\tau(x)) \subset B_k$ and thus $\tau(x) < 2R_k$. Then, using (4.3), we get $\tau(b_k) \le \tau(x) + \frac{1}{8}|x - b_k| < \frac{17}{8} R_k$. Then, each B_k is a $\frac{8}{17}$-super-critical interval for τ and Corollary 2.10 implies

$$\int_{\frac{x}{2}}^{2x} e^{-\frac{|y-x|^2}{4s}} |f(y)| dy \lesssim \|f\|_{BMO_\tau(\omega)} \left(\omega(B_{k_0}) + \sum_{k=k_0}^{k_1-1} e^{-c2^{2k}} \omega(B_{k+1}) \right).$$

On the other hand, since $B_k \subset (\frac{x}{2}, 2x)$, they are local intervals and by (2.3) we have $\omega(B_k) \lesssim |B_k| \inf_{y \in B_k} \omega(y) \lesssim 2^k \sqrt{s}\, \omega(x)$, since x is a Lebesgue point of ω contained in B_k. This gives us

$$\begin{aligned} \frac{1}{\sqrt{s}} \int_{\frac{x}{2}}^{2x} e^{-\frac{|y-x|^2}{4s}} |f(y)| dy &\lesssim \|f\|_{BMO_\tau(\omega)} \omega(x) \left(2^{k_0} + \sum_{k=k_0}^{k_1 - 1} e^{-c2^{2k}} 2^{k+1} \right) \\ &\lesssim \|f\|_{BMO_\tau(\omega)} \omega(x). \end{aligned}$$

Therefore,

$$\mathrm{T}^*_{loc,\tau} f(x) \lesssim \|f\|_{BMO_\tau(\omega)} \omega(x)$$

holds for $a.e.$ $x \in \mathbb{R}^+$ and that completes the proof. □

4 Weighted BMO spaces associated to the Laguerre functions $\{\varphi_n^\alpha\}$.

We consider now the heat diffusion semigroup associated to the Laguerre functions $\{\varphi_n^\alpha\}$ given by (1.1), where $\alpha \geq -1/2$, and its associated maximal operator, $\mathrm{W}^*_{\varphi^\alpha}$. In [7], Dziubański defined in this context the Hardy Space

$$H^1_{L^\alpha} = \{f \in L^1(\mathbb{R}^+) : \mathrm{W}^*_{\varphi^\alpha} f \in L^1(\mathbb{R}^+)\}, \tag{4.1}$$

providing an atomic decomposition. The intervals related to the atoms of $H^1_{L^\alpha}$ were asked to satisfy different conditions, according to a critical radius function:

$$\rho(x) = \frac{1}{8} \min\{x, \frac{1}{x}\}. \tag{4.2}$$

From that it seems reasonable to introduce as suitable BMO weighted space the $BMO_\rho(\omega)$ associated to the critical radius function ρ given by (4.2).

It is a straightforward verification to check that ρ satisfies

$$\rho(y) \leq \rho(x) + \frac{1}{8}|x - y|. \tag{4.3}$$

for all x and y in $\mathbb{R}^+$, which is condition (2.6) for $\gamma = 1/8$. Therefore, according to Proposition 2.2, any sub-critical interval for ρ is also a $\frac{9}{7}$-local interval.

In this section we will prove that the operator $\mathrm{W}^*_{\varphi^\alpha}$ preserves the $BMO_\rho(\omega)$ spaces for any $\alpha \geq -1/2$, under appropriate assumptions on ω.

First we remind that, as it was shown in [14] and [3], the semigroup maximal operator can be expressed as

$$\mathrm{W}^*_{\varphi^\alpha} f(x) = \sup_{0<s<1} \left| \int_0^\infty W_{\varphi^\alpha}(s, x, y) f(y) dy \right|,$$

where the kernel $W_{\varphi^\alpha}(s, x, y)$ is given by

$$W_{\varphi^\alpha}(s, x, y) = \frac{1 - s^2}{2s} (xy)^{\frac{1}{2}} e^{-\frac{1}{4}(s + \frac{1}{s})(x^2 + y^2)} I_\alpha \left(\frac{1 - s^2}{2s} xy \right). \tag{4.4}$$

Here, $I_\alpha(z) = e^{-i\alpha\pi} J_\alpha(iz)$ is the modified Bessel function (J_α being the usual Bessel function, see [9]). We will be using the following estimates for I_α. For a proof see, e.g., [15].

Lemma 4.1. *For a given $\alpha > -1$ we have*

a.

$$I_\alpha(z) \simeq z^\alpha, \text{ for any } 0 < z \le 1,$$

b.

$$I_\alpha(z) \simeq z^{-1/2} e^z, \text{ for any } z \ge 1$$

and

c.

$$|\sqrt{2\pi z}\, e^{-z} I_\alpha(z) - 1| \lesssim \frac{1}{z}, \qquad \text{for every } z \ge \frac{1}{8}.$$

As we have proved in [3], the following estimates of the kernel $W_{\varphi^\alpha}(s,x,y)$ will be useful in the proof of the Theorem.

Lemma 4.2. *For the kernel $W_{\varphi^\alpha}(s,x,y)$ given by (4.4), with $\alpha \ge -\frac{1}{2}$, we have that*

$$W_{\varphi^\alpha}(s,x,y) \lesssim \left(\frac{x^2}{s}\right)^{\alpha+1} e^{-\frac{1}{16}\frac{x^2}{s}} \frac{1}{x^{\alpha+\frac{3}{2}}} y^{\alpha+\frac{1}{2}},$$

for any $0 < s < 1$ and $0 < y < \frac{x}{2}$.

Proof. Using Lemma 4.1 a. in (4.4) we have, for $0 < \frac{1-s^2}{2s} xy \le 1$, that

$$\begin{aligned} W_{\varphi^\alpha}(s,x,y) &\simeq \left(\frac{1-s^2}{2s}\right)^{\alpha+1} (xy)^{\alpha+\frac{1}{2}} e^{-\frac{1}{4}(s+\frac{1}{s})(x^2+y^2)} \\ &\lesssim \left(\frac{x^2}{s}\right)^{\alpha+1} e^{-\frac{1}{4}\frac{x^2}{s}} \frac{1}{x^{\alpha+\frac{3}{2}}} y^{\alpha+\frac{1}{2}}. \end{aligned}$$

On the other hand, if $\frac{1-s^2}{2s} xy \ge 1$, using Lemma 4.1 b. we obtain

$$W_{\varphi^\alpha}(s,x,y) \lesssim \frac{1-s^2}{2s} (xy)^{\frac{1}{2}} \left(\frac{1-s^2}{2s} xy\right)^{-\frac{1}{2}} e^{-\frac{1}{4}(s+\frac{1}{s})(x^2+y^2)} e^{\frac{1-s^2}{2s} xy}.$$

Since $\frac{1-s^2}{2s} xy \ge 1$ and $\alpha \ge -\frac{1}{2}$, we have

$$\left(\frac{1-s^2}{2s} xy\right)^{-\frac{1}{2}} \le \left(\frac{1-s^2}{2s} xy\right)^{\alpha}.$$

On the other hand

$$\begin{aligned} e^{-\frac{1}{4}(s+\frac{1}{s})(x^2+y^2)} e^{\frac{1-s^2}{2s} xy} &= e^{-\frac{s}{4}(x+y)^2} e^{-\frac{1}{4s}(x-y)^2} \\ &\le e^{-\frac{1}{16}\frac{x^2}{s}}, \end{aligned}$$

where in the last inequality we have used $|x-y|>x/2$, since $0<y<\frac{x}{2}$. This gives us

$$W_{\varphi^\alpha}(s,x,y)\lesssim\left(\frac{x^2}{s}\right)^{\alpha+1}e^{-\frac{1}{16}\frac{x^2}{s}}\frac{1}{x^{\alpha+\frac{3}{2}}}y^{\alpha+\frac{1}{2}}.$$

Thus, we have obtained the desired estimate. $\square$

Next, we state the boundedness of $\mathrm{W}^*_{\varphi^\alpha}$ over $BMO_\rho(\omega)$ under appropriate conditions on the weight.

For a given $\eta>-1/2$ and $\theta\geq 0$, consider the class $A_1^{\eta,\theta}$ of those weights ω that satisfy

$$\int_I\omega(x)x^\eta dx\quad\sup_{x\in I}\omega^{-1}(x)x^\eta\leq C\left(\frac{1+b}{1+a}\right)^\theta\int_I x^{2\eta}dx,\tag{4.5}$$

for any interval $I=(a,b)\subset\mathbb{R}^+$. Here, by "sup" we mean the essential supremum with respect to the Lebesgue measure. When $\theta=0$, we denote the class with A_1^η.

It is immediate to check that these weights are in particular in A^1_{loc}. Further, $\omega\in A_1^{\eta,\theta}$ implies that $\omega(x)x^{-\eta}(1+x)^{\frac{\theta}{2}}$ belongs to $A^1(d\mu(x))$, the usual A^1 class on $\mathbb{R}^+$ with measure $d\mu(x)=\frac{x^{2\eta}}{(1+x)^\theta}dx$. Also let us notice that the classes $A_1^{\eta,\theta}$ are increasing with θ. We denote $A_1^{\eta,\infty}\doteq\bigcup_{\theta>0}A_1^{\eta,\theta}$.

Regarding power weights, easy computations show that $x^\delta\in A_1^{\eta,\infty}$ if and only if $-\eta-1<\delta\leq\eta$, that is we get the same powers weights that belong to A_1^η. For weights of the form $\omega(x)=(1+x)^\delta$, which behave like a constant for $0<x<1$ and like x^δ for $x>1$, we have that $\omega\in A_1^\eta$ only if $\eta\geq 0$ and $-\eta-1<\delta\leq\eta$. However, such weights belong to $A_1^{\eta,\infty}$ for any $\delta\in\mathbb{R}$, provided that $\eta\geq 0$.

Now we are ready to state the main theorem of this section.

Theorem 4.3. *Let $\alpha\geq-1/2$. If a weight ω belongs to $A_1^{\alpha+\frac{1}{2},\infty}$, then $\mathrm{W}^*_{\varphi^\alpha}$ is bounded on $BMO_\rho(\omega)$.*

As an immediate consequence of the above result we get the following statement for power weights.

Corollary 4.4. *For $\alpha\geq-\frac{1}{2}$ and a power weight $\omega(x)=x^\delta$, we have that $\mathrm{W}^*_{\varphi^\alpha}$ is bounded on $BMO_\rho(\omega)$ if $-\alpha-\frac{3}{2}<\delta\leq\alpha+\frac{1}{2}$.*

Let us point out that the above intervals for the power δ coincide with the limiting case $p=\infty$ given in Theorem 2.2 of [3], which were shown to be optimal. To check that, it is need to replace the exponent δ by $-\delta p$ in the theorem, and then let p tend to infinity.

Proof of Theorem 4.3. For T^*_{loc}, the local classic heat maximal operator, we can write $\mathrm{W}^*_{\varphi^\alpha}=\mathrm{T}^*_{loc}+(\mathrm{W}^*_{\varphi^\alpha}-\mathrm{T}^*_{loc})$. According to Theorem 3.4 with $\tau=\rho$, we only need to prove that $\mathrm{W}^*_{\varphi^\alpha}-\mathrm{T}^*_{loc}$ is bounded on $BMO_\rho(\omega)$. In fact we shall prove the stronger inequality

$$|\mathrm{W}^*_{\varphi^\alpha}f(x)-\mathrm{T}^*_{loc}f(x)|\lesssim\|f\|_{BMO_\rho(\omega)}\omega(x),\quad\text{for a. e. }x\in\mathbb{R}^+,\tag{4.6}$$

that is, $\|(\mathrm{W}^*_{\varphi^\alpha} - \mathrm{T}^*_{loc})f\|_{L^\infty(\omega^{-1})} \lesssim \|f\|_{BMO_\rho(\omega)}$.

Let $x \in \mathbb{R}^+$ be a Lebesgue point of ω. Then, for $f \in BMO_\rho(\omega)$, we split $|\mathrm{W}^*_{\varphi^\alpha} f(x) - \mathrm{T}^*_{loc} f(x)|$ into four parts:

$$|\mathrm{W}^*_{\varphi^\alpha} f(x) - \mathrm{T}^*_{loc} f(x)| \leq If(x) + IIf(x) + IIIf(x) + IVf(x),$$

where

$$If(x) = \left| \mathrm{W}^*_{\varphi^\alpha} f(x) - \sup_{0<s<1} \left| \int_{\frac{x}{2}}^{2x} W_{\varphi^\alpha}(s,x,y) f(y)dy \right| \right|,$$

$$IIf(x) = \sup_{\rho(x)^2 \leq s < 1} \left| \int_{\frac{x}{2}}^{2x} W_{\varphi^\alpha}(s,x,y) f(y)dy \right|,$$

$$IIIf(x) = \sup_{0<s<\rho(x)^2} \int_{\frac{x}{2}}^{2x} |W_{\varphi^\alpha}(s,x,y) - T_s(x,y)||f(y)|dy,$$

and

$$IVf(x) = \left| \sup_{0<s<\rho(x)^2} \left| \int_{\frac{x}{2}}^{2x} T_s(x,y) f(y)dy \right| - \mathrm{T}^*_{loc} f(x) \right|.$$

So we have to prove estimate (4.6) for each term.

For the term $If(x)$, observe that

$$\begin{aligned} If(x) &\leq \sup_{0<s<1} \int_0^{\frac{x}{2}} W_\alpha(s,x,y)|f(y)|dy + \sup_{0<s<1} \int_{2x}^{\infty} W_\alpha(s,x,y)|f(y)|dy \\ &\doteq A_0 f(x) + A_\infty f(x). \end{aligned}$$

Using the estimate of Lemma 4.2 we have

$$\begin{aligned} A_0 f(x) &\leq C \sup_{0<s<1} \left(\frac{x^2}{s}\right)^{\alpha+1} e^{-\frac{1}{16}\frac{x^2}{s}} \frac{1}{x^{\alpha+\frac{3}{2}}} \int_0^x |f(y)| y^{\alpha+\frac{1}{2}} dy \\ &\leq C_{\alpha,\epsilon}\, e^{-\epsilon x^2} \frac{1}{x^{\alpha+\frac{3}{2}}} \int_0^x |f(y)| y^{\alpha+\frac{1}{2}} dy, \end{aligned}$$

for some $\epsilon > 0$.

Since intervals of the form $(2^{-i-1}x, 2^{-i}x)$ for any integer i are super-critical for ρ, we have

$$\begin{aligned}
\int_0^x |f(y)| y^{\alpha+\frac{1}{2}} dy &\lesssim \sum_{i=0}^{\infty} (2^{-i}x)^{\alpha+\frac{1}{2}} \int_{2^{-i-1}x}^{2^{-i}x} |f(y)| dy \\
&\lesssim \|f\|_{BMO_\rho(\omega)} \sum_{i=0}^{\infty} (2^{-i}x)^{\alpha+\frac{1}{2}} \int_{2^{-i-1}x}^{2^{-i}x} \omega(y) dy \\
&\lesssim \|f\|_{BMO_\rho(\omega)} \int_0^x \omega(y) y^{\alpha+\frac{1}{2}} dy \\
&\lesssim \|f\|_{BMO_\rho(\omega)} (1+x)^\theta x^{2\alpha+2} \inf_{(0,x)} \omega(y) y^{-\alpha-\frac{1}{2}} \\
&\lesssim \|f\|_{BMO_\rho(\omega)} \omega(x) x^{\alpha+\frac{3}{2}} (1+x)^\theta .
\end{aligned}$$

Therefore,

$$A_0 f(x) \lesssim \|f\|_{BMO_\rho(\omega)} e^{-\epsilon x^2} \omega(x)(1+x)^\theta \lesssim \|f\|_{BMO_\rho(\omega)} \omega(x).$$

To take care of A_∞ term we first check that a weight in our class satisfies the inequality

$$b^{-\eta} \inf_{y \in (\frac{b}{2}, b)} \omega(y) \le C \left(\frac{1+b}{1+a} \right)^\theta \inf_{y \in (a, 2a)} \omega(y) y^{-\eta}, \tag{4.7}$$

for any positive a and b such that $b \ge 2a$. Indeed,

$$\begin{aligned}
b^{-\eta} \inf_{y \in (\frac{b}{2}, b)} \omega(y) &\le b^{-\eta-1} \int_{\frac{b}{2}}^b \omega(y) dy \\
&\le C b^{-2\eta-1} \int_a^b \omega(y) y^\eta dy.
\end{aligned}$$

Now we use (4.5) for the interval (a, b) to obtain

$$\begin{aligned}
b^{-\eta} \inf_{y \in (\frac{b}{2}, b)} \omega(y) &\le C \left(\frac{1+b}{1+a} \right)^\theta b^{-2\eta-1} \inf_{y \in (a,b)} \omega(y) y^{-\eta} \int_a^b y^{2\eta} dy \\
&\le C \left(\frac{1+b}{1+a} \right)^\theta \inf_{y \in (a, 2a)} \omega(y) y^{-\eta}.
\end{aligned}$$

Next, using the symmetry of the kernel (4.4) and Lemma 4.2, we can estimate $A_\infty f(x)$ by

$$A_\infty f(x) \quad \lesssim \sup\nolimits_{0<s<1} x^{\alpha+\frac{1}{2}} \int_{2x}^\infty \left(\tfrac{y^2}{s} \right)^{\alpha+1} e^{-\epsilon \frac{y^2}{s}} \tfrac{1}{y^{\alpha+\frac{3}{2}}} |f(y)| dy$$

for some positive constant ϵ. Now, for a fixed s with $0 < s < 1$ we consider the intervals $J_{k,s} \doteq (2^k \sqrt{s}, 2^{k+1} \sqrt{s})$ for $k \ge k_0$, where k_0 denotes the integer such that $2^{k_0} \sqrt{s} < x \le 2^{k_0+1} \sqrt{s}$. Note that $J_{k,s}$ are 1/3-supercritical for ρ and also local intervals. Thus we have

$$\begin{aligned}
\int_x^\infty \left(\frac{y^2}{s}\right)^{\alpha+1} e^{-\epsilon\frac{y^2}{s}}|f(y)|\frac{1}{y^{\eta+1}}dy &\leq \sum_{k\geq k_0}\int_{J_{k,s}} \left(\frac{y^2}{s}\right)^{\alpha+1} e^{-\epsilon\frac{y^2}{s}}|f(y)|\frac{1}{y^{\eta+1}}dy \\
&\leq C\sum_{k\geq k_0}\frac{4^{k(\alpha+1)}e^{-\epsilon 4^k}}{(2^k\sqrt{s})^{\eta+1}}\int_{J_{k,s}}|f(y)|dy \\
&\leq C\|f\|_{BMO_\rho(\omega)}\sum_{k\geq k_0}\frac{4^{k(\alpha+1)}e^{-\epsilon 4^k}}{(2^k\sqrt{s})^{\eta}}\frac{\omega(J_{k,s})}{|J_{k,s}|} \\
&\leq C\|f\|_{BMO_\rho(\omega)}\sum_{k\geq k_0}\frac{4^{k(\alpha+1)}e^{-\epsilon 4^k}}{(2^k\sqrt{s})^{\eta}}\inf_{y\in J_{k,s}}\omega(y),
\end{aligned}$$

where in the last inequality we have used (2.3), since $\omega \in A^1_{loc}$ and J_k is a 2-local interval.

By our choice of k_0, we have that $x \leq 2^{k+1}\sqrt{s}$ for any $k \geq k_0$. Then, using (4.7) with $b = 2^{k+1}\sqrt{s}$ and $a = \frac{x}{2}$ (observe that $2a \leq b$) we obtain, for any $k \geq k_0$, that

$$\begin{aligned}
(2^k\sqrt{s})^{-\eta}\inf_{y\in J_k}\omega(y) &\leq C\inf_{y\in(\frac{x}{2},x)}\omega(y)y^{-\eta} \\
&\leq C\omega(x)x^{-\eta}.
\end{aligned}$$

Since the series $\sum_{-\infty}^{\infty} 4^{k(\alpha+1)}e^{-\frac{\epsilon}{2}4^k}$, being $\alpha+1>0$, is convergent we get that the desired inequality (4.6) holds also for A_∞.

Now, let us note that $W_{\varphi^\alpha}(s,x,y) \lesssim T_s(x,y)$, for any $0<s<1$ and $0<x,y<\infty$. This follows easily using the estimates for the Bessel functions of Lemma 4.1. Then, both $IIf(x)$ and $IVf(x)$ are controlled by $\mathrm{T}^*_{loc,\rho}|f|(x)$, where this operator have been defined in (3.7). Since $f\in BMO_\rho(\omega)$ implies $|f|\in BMO_\rho(\omega)$, Proposition 3.5 gives us

$$IIf(x)+IV(x) \leq C\|f\|_{BMO_\rho(\omega)}\omega(x).$$

Finally, we consider the third term $IIIf(x)$. Let us rewrite the kernel (4.4) as

$$W_{\varphi^\alpha}(s,x,y) = \Phi_{clas}(s,x,y)\Phi^\alpha_{bes}(s,x,y)\Phi_{err}(s,x,y),$$

where

$$\Phi_{clas}(s,x,y) = \left(\frac{4\pi s}{1+s^2}\right)^{-\frac{1}{2}} e^{-\frac{1+s^2}{4s}|x-y|^2}, \tag{4.8}$$

$$\Phi^\alpha_{bes}(s,x,y) = \left(2\pi\frac{1-s^2}{2s}xy\right)^{\frac{1}{2}} e^{-\frac{1-s^2}{2s}xy} I_\alpha\left(\frac{1-s^2}{2s}xy\right) \tag{4.9}$$

and

$$\Phi_{err}(s,x,y) = \left(\frac{1-s^2}{1+s^2}\right)^{\frac{1}{2}} e^{-sxy}. \tag{4.10}$$

Let us write

$$
\begin{aligned}
|W_{\varphi^\alpha}(s,x,y)-T_s(x,y)| &= |\Phi_{clas}(s,x,y)\Phi^\alpha_{bes}(s,x,y)\Phi_{err}(s,x,y)-T_s(x,y)|\\
&\leq |\Phi_{clas}(s,x,y)-T_s(x,y)|\Phi^\alpha_{bes}(s,x,y)\Phi_{err}(s,x,y)\\
&+ |\Phi^\alpha_{bes}(s,x,y)-1|T_s(x,y)\Phi_{err}(s,x,y)\\
&+ |\Phi_{err}(s,x,y)-1|T_s(x,y)\\
&\doteq \sum_{i=1}^{3}\Omega_i(s,x,y).
\end{aligned}
$$

We want to prove that

$$
\int_{\frac{x}{2}}^{2x}\Omega_i(s,x,y)|f(y)|dy \leq \|f\|_{BMO_\rho(\omega)}\omega(x), \tag{4.11}
$$

for any $0<s<\rho(x)^2$ and $i=1,2,3$.

Consider first $\Omega_1(s,x,y)$. If we take, for fixed x and y, the function $h(t)=(4\pi t)^{-1/2}e^{-\frac{|x-y|^2}{4t}}$, we have, from (1.7) and (4.8), that $\Phi_{clas}(s,x,y)=h\left(\frac{s}{1+s^2}\right)$ and $T_s(x,y)=h(s)$. Then, the mean value theorem for h in $[s,\frac{2s}{1+s^2}]$ implies

$$
|\Phi_{clas}(s,x,y)-T_s(x,y)|\leq C\ s^{3/2},
$$

with C independent of x and y. Also, using that $\Phi^\alpha_{bes}(s,x,y)\leq C$ and $\Phi_{err}(s,x,y)\leq e^{-\frac{1}{2}sx^2}$ for $\frac{x}{2}\leq y\leq 2x$ and $0<s<1$, we have

$$
\Omega_1(s,x,y)\leq C\frac{1}{x}. \tag{4.12}
$$

Setting $I_x=(\frac{x}{2},2x)$ and noting that I_x is super-critical for ρ and also a 4-local interval, we obtain

$$
\int_{\frac{x}{2}}^{2x}\Omega_1(s,x,y)|f(y)|dy\leq C\frac{1}{|I_x|}\int_{I_x}|f(y)|dy\leq C\|f\|_{BMO_\rho(\omega)}\inf_{y\in I_x}\omega(y).
$$

Therefore (4.11) holds for $i=1$ using that x is a Lebesgue point of ω.

On the other hand, we have

$$
\Omega_2(s,x,y)\leq C|\Phi^\alpha_{bes}(s,x,y)-1|s^{-\frac{1}{2}}.
$$

Since $\frac{x}{2}<y<2x$ and $0<s<\rho(x)^2$, which implies $0<s<\frac{1}{64}$ and $0<s<\frac{x^2}{64}$, it is not difficult to check that $\frac{1-s^2}{2s}xy>\frac{1}{8}$ (more precisely, we obtain $\frac{1-s^2}{2s}xy>\frac{63}{4}$). Then, from (4.9) and Lemma 4.1 c., we obtain (4.12) for $\Omega_2(s,x,y)$ and hence (4.11).

Consider now $i=3$. We can write

$$
\begin{aligned}
\Omega_3(s,x,y) &= |\Phi_{err}(s,x,y)-1|T_s(x,y)\\
&\leq |\left(\frac{1-s^2}{1+s^2}\right)^{1/2}-1|e^{-sxy}T_s(x,y)+|e^{-sxy}-1|T_s(x,y)\\
&\doteq \Omega_{31}(s,x,y)+\Omega_{32}(s,x,y).
\end{aligned}
$$

For the first of those kernels, since $y \simeq x$ and $s < \frac{x^2}{64}$, we get again an estimate like (4.12):

$$\Omega_{31}(s,x,y) \lesssim s^{3/2} e^{-\frac{1}{2}sx^2} \lesssim \frac{1}{x}.$$

Finally, for the kernel $\Omega_{32}(s,x,y)$, by the mean value theorem and using $y \simeq x$ we have

$$\Omega_{32}(s,x,y) \lesssim \sqrt{s}x^2 e^{-\frac{|y-x|^2}{4s}}.$$

If $0 < x \leq 1$, then $\sqrt{s}x^2 \leq 1 \leq \frac{1}{x}$ and we get that also $\Omega_{32}(s,x,y) \leq \frac{1}{x}$.

Consider now $x > 1$. Since $\sqrt{s} < \rho(x) = \frac{1}{8x}$, we have $\sqrt{s}x^2 < x$. Therefore

$$\begin{aligned} \int_{\frac{x}{2}}^{2x} \Omega_{32}(s,x,y)|f(y)|dy &\lesssim x \int_{\frac{x}{2}}^{x-\frac{1}{8x}} e^{-\frac{|y-x|^2}{4s}} |f(y)|dy \\ &+ \frac{1}{\rho(x)} \int_{B(x,\rho(x))} |f(y)|dy \\ &+ x \int_{x+\frac{1}{8x}}^{2x} e^{-\frac{|y-x|^2}{4s}} |f(y)|dy. \end{aligned} \tag{4.13}$$

Since $f \in BMO_\rho(\omega)$, $\omega \in A^1_{loc}$ and x is a Lebesgue point of ω, we can bound the term in the middle by a constant times $\|f\|_{BMO_\rho(\omega)}\omega(x)$.

For the first term we can write

$$x \int_{\frac{x}{2}}^{x-\frac{1}{8x}} e^{-\frac{|y-x|^2}{4s}} |f(y)|dy \leq \sum_{k=1}^{k_0} x \int_{x-\frac{k+1}{8x}}^{x-\frac{k}{8x}} e^{-\frac{|y-x|^2}{4s}} |f(y)|dy, \tag{4.14}$$

where k_0 is an integer that satisfies $x - \frac{k_0+1}{8x} \leq \frac{x}{2} < x - \frac{k_0}{8x}$. That is, we choose k_0 such that $k_0 < 4x^2 \leq k_0 + 1$. Observe that $x \geq 1$ implies $k_0 \geq 3$.

If $y \in (x - \frac{k+1}{8x}, x - \frac{k}{8x})$, then $|x-y|^2 > \frac{k^2}{64x^2}$, and since $s < \rho(x)^2 = \frac{1}{64x^2}$, we obtain

$$e^{-\frac{|y-x|^2}{4s}} \leq e^{-\frac{k^2}{4}}.$$

Thus, if we call $I_x^k \doteq B(x, \frac{k+1}{8x})$,we have

$$x \int_{x-\frac{k+1}{8x}}^{x-\frac{k}{8x}} e^{-\frac{|y-x|^2}{4s}} |f(y)|dy \lesssim \frac{k}{|I_x^k|} e^{-\frac{k^2}{4}} \int_{I_x^k} |f(y)|dy. \tag{4.15}$$

For $1 \leq k \leq k_0$, observe that each I_x^k is a critical or super-critical interval, since $\frac{k+1}{8x} \geq \rho(x)$. Also, $k_0 < 4x^2$ implies $I_x^k \subset (\frac{3}{8}x, \frac{13}{8}x)$, and hence I_x^k are local intervals. Then, (4.15) is bounded by a constant times $\|f\|_{BMO_\rho(\omega)}\, \omega(x) k\, e^{-\frac{k^2}{4}}$. Plugging this estimate into (4.14), we obtain the desired inequality for the first term of (4.13).

Finally, for the third term of (4.13), we choose an integer $k_1 \geq 1$ such that $x + \frac{k_1}{8x} < 2x \leq x + \frac{k_1+1}{8x}$ (or equivalently, $k_1 < 8x^2 \leq k_1 + 1$) and, analogously as we did with the first term, we obtain

$$\begin{aligned} x\int_{x+\frac{1}{8x}}^{2x} e^{-\frac{|y-x|^2}{4s}}|f(y)|dy &\leq x\sum_{k=1}^{k_1}\int_{x+\frac{k}{8x}}^{x+\frac{k+1}{8x}} e^{-\frac{|y-x|^2}{4s}}|f(y)|dy \\ &\leq x\sum_{k=1}^{k_1} e^{-\frac{k^2}{4}}\int_{J_x^k}|f(y)|dy, \end{aligned}$$

where $J_x^k = (x-\frac{1}{8x}, x+\frac{k+1}{8x})$. Let us call c_k and R_k to the center and the radius of J_x^k, respectively. Observe that the center of J_x^0 is x. Then, for $k \geq 1$, we have $c_k \geq x$ and since ρ is decreasing on $(1,\infty)$, we obtain $\rho(c_k) \leq \rho(x) = \frac{1}{8x} < (\frac{k}{2}+1)\frac{1}{8x} = R_k$. Therefore, J_x^k is a super-critical interval for ρ and since it is also local we may proceed as above arriving to the same estimate.

Altogether we get

$$\int_{\frac{x}{2}}^{2x} \Omega_{32}(s,x,y)|f(y)|dy \leq C\|f\|_{BMO_\rho(\omega)}\omega(x).$$

The proof of the Theorem is now complete. □

5 Weighted BMO spaces associated to the Laguerre functions $\{\mathcal{L}_n^\alpha\}$.

In this section we introduce BMO_σ, the BMO spaces related to the Laguerre functions $\{\mathcal{L}_n^\alpha\}$. We will prove the boundedness over those spaces of the maximal operator using the relationship between the systems $\{\mathcal{L}_n^\alpha\}$ and $\{\varphi_n^\alpha\}$.

For the family of Laguerre functions $\{\mathcal{L}_n^\alpha\}$ given by (1.2), we denote by $\mathrm{W}^*_{\mathcal{L}^\alpha}$ the Maximal Operator associated to this semigroup. As in the case of the Laguerre functions $\{\varphi_n^\alpha\}$ and the operator $\mathrm{W}^*_{\varphi^\alpha}$, in [7], Dziubański also considered the Hardy type space

$$H^1_{\mathcal{L}^\alpha} = \{f \in L^1 : \mathrm{W}^*_{\mathcal{L}^\alpha} f \in L^1\},$$

providing an atomic decomposition.

The suitable weighted BMO spaces for the systems $\{\mathcal{L}_n^\alpha\}$ arises from Definition 2 when the critical radius function is $\sigma(x) = \frac{1}{8}\min\{x,1\}$. It is not difficult to prove that σ satisfies the critical radius condition (2.6) with $\gamma = 1/8$, that is

$$\sigma(y) \leq \sigma(x) + \frac{1}{8}|x-y|, \tag{5.1}$$

for any x and y in $\mathbb{R}^+$. The space BMO_σ, for $\omega \equiv 1$, is actually the dual space of $H^1_{\mathcal{L}^\alpha}$, introduced by Dziubański, when $\alpha > 0$.

Next, for $\alpha \geq 0$, we shall prove boundedness results for $\mathrm{W}^*_{\mathcal{L}^\alpha}$ on the spaces $BMO_\sigma(\upsilon)$ under suitable conditions over the weight υ.

Theorem 5.1. *Let $\alpha \geq 0$ and υ a weight belonging to $A_1^{\frac{\alpha}{2},\infty}$, then $\mathrm{W}^*_{\mathcal{L}^\alpha}$ is bounded on $BMO_\sigma(\upsilon)$.*

Corollary 5.2. *For $\alpha \geq 0$ and a power weight $\omega(x) = x^\delta$, we have that $\mathrm{W}^*_{\mathcal{L}^\alpha}$ is bounded on $BMO_\sigma(\upsilon)$ if $-\frac{\alpha}{2} - 1 < \delta \leq \frac{\alpha}{2}$.*

For the proof of this theorem, let us note that, from definitions (1.1) and (1.2), there exists a relationship between the Laguerre functions φ_n^α and $\mathcal{L}_n^\alpha$, namely

$$\mathcal{L}_n^\alpha(x) = \frac{1}{\sqrt{2}}\varphi_n^\alpha(x^{\frac{1}{2}})x^{-\frac{1}{4}}. \tag{5.2}$$

In [3] it has been shown that $\mathrm{W}^*_{\mathcal{L}^\alpha}$ can be expressed as

$$\mathrm{W}^*_{\mathcal{L}^\alpha} g(x) = \sup_{0<s<1} \left| \int_0^\infty W_{\mathcal{L}^\alpha}(s, x, y) g(y) dy \right|,$$

where

$$W_{\mathcal{L}^\alpha}(s, x, y) = \frac{1}{2}(xy)^{-\frac{1}{4}} W_{\varphi^\alpha}(s, \sqrt{x}, \sqrt{y}). \tag{5.3}$$

This equality suggests that the result for $\mathrm{W}^*_{\mathcal{L}^\alpha}$ of Theorem 5.1 could be derived from the analogous one for $\mathrm{W}^*_{\varphi^\alpha}$ (see Theorem 4.3).

Based on (5.2), we define a linear transformation R acting on measurable functions defined on $\mathbb{R}^+ = (0, \infty)$ as follows

$$\mathrm{R}f(x) = \frac{1}{\sqrt{2}} f(x^{\frac{1}{2}})x^{-\frac{1}{4}}. \tag{5.4}$$

Clearly, R is an isomorphism in $L^1_{loc}(0, \infty)$ and its inverse is given by

$$\mathrm{R}^{-1}g(y) = \sqrt{2}g(y^2)y^{\frac{1}{2}}. \tag{5.5}$$

For this operator we have the the following transference result.

Proposition 5.3. *A weight ω belongs to A^1_{loc} if and only if $\upsilon = \mathrm{R}\omega$ belongs to A^1_{loc}. Moreover, R is an isomorphism between the Banach spaces $BMO_\rho(\omega)$ and $BMO_\sigma(\upsilon)$, provided $\omega \in A^1_{loc}$.*

For the proof, we will use the following lemma.

Lemma 5.4. *If (a, b) is a critical interval for σ, then $(\sqrt{a}, \sqrt{b})$ is a $\frac{1}{2}$-super-critical interval for ρ. Conversely, if (a, b) is a critical interval for ρ, then (a^2, b^2) is a super-critical interval for σ.*

Proof. Let us remind that $\rho(x) = \frac{1}{8}\min\{x, \frac{1}{x}\}$ and $\sigma(x) = \frac{1}{8}\min\{x, 1\}$. Consider first $I = (a, b)$ a critical interval for σ, that is, $|I| = 2\sigma\left(\frac{a+b}{2}\right)$. Let us call $\widetilde{I} \doteq (\sqrt{a}, \sqrt{b})$. Then

$$|\widetilde{I}| = |I|\frac{1}{\sqrt{a} + \sqrt{b}} = \sigma\left(\frac{a+b}{2}\right)\frac{2}{\sqrt{a} + \sqrt{b}}.$$

If $\frac{a+b}{2} \leq 1$ then

$$\sigma\left(\frac{a+b}{2}\right) = \frac{1}{8}\frac{a+b}{2} \geq \frac{1}{8}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2,$$

and this implies

$$|\widetilde{I}| \geq \frac{1}{8} \frac{\sqrt{a}+\sqrt{b}}{2} \geq \rho\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right).$$

If $\frac{a+b}{2} \geq 1$ then $\sigma\left(\frac{a+b}{2}\right) = \frac{1}{8}$ and then

$$|\widetilde{I}| = \frac{1}{8} \frac{2}{\sqrt{a}+\sqrt{b}} \geq \rho\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right).$$

Therefore, $\widetilde{I}$ is a $\frac{1}{2}$-super-critical interval for ρ.

Consider now $I = (a, b)$ a critical interval for ρ, that is $|I| = 2\rho\left(\frac{a+b}{2}\right)$, and let us call $I' = (a^2, b^2)$ Then

$$|I'| = |I|(a+b) = 2\rho\left(\frac{a+b}{2}\right)(a+b).$$

If $\frac{a+b}{2} \leq 1$ then $\rho\left(\frac{a+b}{2}\right) = \frac{1}{8}\frac{a+b}{2}$ and this implies

$$\begin{aligned} |I'| &= \frac{1}{8}(a+b)^2 \\ &\geq \frac{1}{8}(a^2+b^2) \\ &\geq 2\sigma\left(\frac{a^2+b^2}{2}\right). \end{aligned}$$

If $\frac{a+b}{2} \geq 1$ then $\rho\left(\frac{a+b}{2}\right) = \frac{1}{8}\frac{2}{a+b}$ and then

$$|I'| = \frac{1}{2} \geq 4\sigma\left(\frac{a^2+b^2}{2}\right).$$

Therefore, I' is a super-critical interval for σ.

$\square$

Proof of Proposition 5.3. Let $\omega \in A^1_{loc}$ and $\upsilon = \mathrm{R}\omega$, given by (5.4). Assume $I = (a, b)$ a κ-local interval. Since

$$\upsilon(I) = \frac{1}{\sqrt{2}} \int_a^b \omega(x^{\frac{1}{2}}) x^{-\frac{1}{4}} dx = \sqrt{2} \int_{\sqrt{a}}^{\sqrt{b}} \omega(u) u^{\frac{1}{2}} du,$$

we have that

$$\upsilon(I) \simeq a^{\frac{1}{4}} \omega(\widetilde{I}), \tag{5.6}$$

where $\widetilde{I} \doteq (\sqrt{a}, \sqrt{b})$. Note that $\widetilde{I}$ is a $\sqrt{\kappa}$-local interval. Then, by (2.3),

$$\begin{aligned} \omega(\widetilde{I}) &\leq C_\kappa\, |\widetilde{I}| \inf_{y \in \widetilde{I}} \omega(y) \\ &\leq C_\kappa\, a^{-\frac{1}{2}} |I| \inf_{x \in I} \omega(x^{\frac{1}{2}}), \end{aligned}$$

where we have used $|\widetilde{I}| = \sqrt{b} - \sqrt{a} = \frac{b-a}{\sqrt{b}+\sqrt{a}} \sim a^{-\frac{1}{2}}|I|$. Then, by (5.6),

$$\begin{aligned} \upsilon(I) &\leq C_\kappa |I| \inf_{x \in I} x^{-\frac{1}{4}} \omega(x^{\frac{1}{2}}) \\ &\simeq C_\kappa |I| \inf_{x \in I} \upsilon(x). \end{aligned}$$

Conversely, if $\upsilon \in A^1_{loc}$, we can prove, in an analogous way, that $\omega = R^{-1}\upsilon$, given by (5.5), belongs to A^1_{loc}, using this time that $I \in \mathcal{I}_\kappa$ implies $I' = (a^2, b^2) \in \mathcal{I}_{\kappa^2}$.

Now, consider $\omega \in A^1_{loc}$ and $f \in BMO_\rho(\omega)$. For $\upsilon = R\omega$, we will show that there exists a constant C, independent of f, such that

$$\frac{1}{\upsilon(I)} \int_I |\mathrm{R}f(x)| dx \leq C \|f\|_{BMO_\rho(\omega)}, \tag{5.7}$$

for any I critical or super-critical interval for σ, and

$$\frac{1}{\upsilon(I)} \int_I |\mathrm{R}f(x) - c| dx \leq C \|f\|_{BMO_\rho(\omega)}, \tag{5.8}$$

for some $c = c(f, I)$ and any I sub-critical interval for σ. This will imply $\|Rf\|_{BMO_\sigma(\upsilon)} \lesssim \|f\|_{BMO_\rho(\omega)}$.

First, to prove (5.7), according to Corollary 2.9, it is enough to consider $I = (a, a*)$ a critical interval for σ. Performing the change of variable $u = \sqrt{x}$, since $a < a^* \leq \frac{9}{7}a$, we obtain

$$\begin{aligned} \int_I |\mathrm{R}f(x)| dx &= \frac{1}{\sqrt{2}} \int_a^{a^*} |f(x^{\frac{1}{2}})| x^{-\frac{1}{4}} dx \\ &= \sqrt{2} \int_{\sqrt{a}}^{\sqrt{a^*}} |f(u)| u^{\frac{1}{2}} du \\ &\simeq a^{\frac{1}{4}} \int_{\widetilde{I}} |f(u)| du, \end{aligned}$$

where $\widetilde{I} = (\sqrt{a}, \sqrt{a*})$. Since, by Lemma 5.4, $\widetilde{I}$ is a $\frac{1}{2}$-super- critical interval for ρ, Corollary 2.10 implies

$$\begin{aligned} \int_{\widetilde{I}} |f(u)| du &\lesssim \|f\|_{BMO_\rho(\omega)} \omega(\widetilde{I}) \\ &\lesssim \|f\|_{BMO_\rho(\omega)} \upsilon(I) a^{-\frac{1}{4}}, \end{aligned} \tag{5.9}$$

where the last inequality follows from (5.6), since I is a local interval. Thus, (5.7) holds for a σ-critical interval I.

Now, consider $I = (a, b)$ a sub-critical interval for σ. Let a^* such that $a < b < a^*$, with (a, a^*) a critical interval for σ. We will prove now that (5.8) holds for I and some constant $c = c(f, I)$.

Making the change of variable $u = \sqrt{x}$ and considering $\widetilde{I} = (\sqrt{a}, \sqrt{b})$, we have

$$\begin{aligned} \int_I |\mathrm{R}f(x) - c| dx &= 2 \int_{\widetilde{I}} |\frac{1}{\sqrt{2}} f(u) u^{-\frac{1}{2}} - c| u du \\ &= \sqrt{2} \int_{\widetilde{I}} |f(u) - \sqrt{2} u^{\frac{1}{2}} c| u^{\frac{1}{2}} du \\ &\lesssim \int_{\widetilde{I}} |f(u) - \sqrt{2} a^{\frac{1}{4}} c| u^{\frac{1}{2}} du + |c| \int_{\widetilde{I}} |a^{\frac{1}{4}} - u^{\frac{1}{2}}| u^{\frac{1}{2}} du \\ &\doteq I + II. \end{aligned}$$

If we choose $c = \frac{1}{\sqrt{2}} a^{-\frac{1}{4}} f_{\widetilde{I}}$, by (5.6) we get

$$\begin{aligned} I &\lesssim a^{\frac{1}{4}} \int_{\widetilde{I}} |f(u) - f_{\widetilde{I}}| du \\ &\lesssim \|f\|_{BMO_\rho(\omega)} \upsilon(I). \end{aligned}$$

On the other hand,

$$\begin{aligned} II &\lesssim |c| a^{\frac{1}{4}} (b^{\frac{1}{4}} - a^{\frac{1}{4}}) |\widetilde{I}| \\ &\lesssim (b^{\frac{1}{4}} - a^{\frac{1}{4}}) \int_{\sqrt{a}}^{\sqrt{a*}} |f(x)| dx \\ &\lesssim \|f\|_{BMO_\rho(\omega)} \upsilon((a, a*)) a^{-\frac{1}{4}} (b^{\frac{1}{4}} - a^{\frac{1}{4}}), \end{aligned}$$

where we have used (5.9) for the critical interval $(a, a*)$.

Since $\upsilon \in A^1_{loc}$ and $I \subset (a, a*)$, which is a local interval, by (2.4) we have $\upsilon((a, a*)) \lesssim \frac{a}{|I|} \upsilon(I)$. Also, observe that

$$b - a = (b^{\frac{1}{4}} - a^{\frac{1}{4}})(b^{\frac{1}{4}} + a^{\frac{1}{4}})(b^{\frac{1}{2}} + a^{\frac{1}{2}}) \simeq (b^{\frac{1}{4}} - a^{\frac{1}{4}}) a^{\frac{3}{4}}.$$

Thus, we obtain that II satisfies the desired inequality (5.8).

The proof that R^{-1}, given by (5.5), is bounded from $BMO_\sigma(\upsilon)$ to $BMO_\rho(\omega)$, follows in an analogous way.

$\square$

Proof of Theorem 5.1. By (5.3) we can write

$$\mathrm{W}^*_{\mathcal{L}^\alpha} = \mathrm{R} \circ \mathrm{W}^*_{\varphi^\alpha} \circ \mathrm{R}^{-1}. \tag{5.10}$$

Then, in view of Proposition 5.3 and Theorem 4.3, we only need to prove that a weight υ belongs to $A^{1,\infty}_{\alpha/2}$ if and only if $\omega = \mathrm{R}^{-1}\upsilon$ belongs to $A^{1,\infty}_{\alpha+1/2}$.

In fact, for such pair υ, ω we have for some $\theta \geq 0$

$$\begin{aligned} \int_a^b \omega(y) y^{\alpha+1/2} dy &= \frac{1}{\sqrt{2}} \int_{a^2}^{b^2} \upsilon(z) z^{\alpha/2} dz \\ &\lesssim \left(\frac{1+a^2}{1+b^2}\right)^\theta \inf_{z \in (a^2, b^2)} \upsilon(z) z^{-\alpha/2} \int_{a^2}^{b^2} z^\alpha dz \\ &\lesssim \left(\frac{1+a}{1+b}\right)^{2\theta} \inf_{y \in (a,b)} \omega(y) y^{-\alpha-1/2} \int_a^b y^{2\alpha+1} dy. \end{aligned}$$

Similarly, it follows that υ belongs to $A^{1,\infty}_{\alpha+1/2}$, provided ω is in $A^{1,\infty}_{\alpha+1/2}$. $\square$

6 Weighted BMO spaces associated to the Laguerre functions $\{\ell^\alpha_n\}$.

In this section we consider the weighted BMO spaces related to the Laguerre functions $\{\ell^\alpha_n\}$ given by (1.3). We will prove the boundedness over those spaces of the maximal operator using the results of the previous section.

In view of the equality $\ell_n^\alpha(x) = \mathcal{L}_n^\alpha(x)x^{-\alpha/2}$, which is evident from definitions (1.2) and (1.3), we may derive also a point-wise relationship between the kernels of both semigroups and write for any measurable non-negative function f

$$\begin{aligned} W^*_{\ell^\alpha} f(x) &= \sup_{0<s<1} x^{-\alpha/2} \int_0^\infty y^{-\alpha/2} W_{\mathcal{L}^\alpha}(s,x,y) f(y) y^\alpha dy \\ &= x^{-\alpha/2} \sup_{0<s<1} \int_0^\infty W_{\mathcal{L}^\alpha}(s,x,y)\, y^{\alpha/2} f(y) dy. \end{aligned} \tag{6.1}$$

Regarding the appropriate BMO space for this system, the critical radius function is the same as in the previous section, namely, $\sigma(x) = \frac{1}{8}\min\{x,1\}$. However, the class of weights will be different, since the system $\{\ell_n^\alpha\}$ is orthonormal with respect to the measure μ, with $d\mu = x^\alpha dx$.

For a measure μ on $\mathbb{R}^+$ let us introduce the more general classes $A_\eta^{1,\infty}(d\mu)$ as those weights for which there exists $\theta \geq 0$ such that the inequality

$$\int_I \omega(x)x^\eta d\mu \quad \sup_{x\in I} \omega^{-1}(x)x^\eta \leq C\left(\frac{1+b}{1+a}\right)^\theta \int_I x^{2\eta} d\mu \tag{6.2}$$

holds for any interval $I = (a,b) \subset \mathbb{R}^+$. Here, with "sup" we denote the essential supremum with respect to the measure μ.

Let us notice that when μ is the Lebesgue measure we obtain the classes $A_\eta^{1,\infty}$ previously defined and that, as before, weights belonging to these classes are in $A^1_{loc}(d\mu)$.

With this notation we are ready to state the result concerning the boundedness of $W^*_{\ell^\alpha}$.

Theorem 6.1. *Let $\alpha \geq 0$ and ω a weight belonging to $A_0^{1,\infty}(x^\alpha dx)$, that is, for some $\theta \geq 0$ there is a constant such that the inequality*

$$\int_I \omega(x)x^\alpha dx \quad \sup_{x\in I} \omega^{-1}(x) \leq C\left(\frac{1+b}{1+a}\right)^\theta \int_I x^\alpha dx \tag{6.3}$$

holds for any interval $I = (a,b)$ contained in $(0,\infty)$. Then, the maximal operator $W^*_{\ell^\alpha}$ is bounded on $BMO_\sigma(\omega)$.

Proof. We first observe that if for ω satisfying (6.3) we set $\upsilon(x) = \omega(x)x^{\alpha/2}$, then υ belongs to $A_{\alpha/2}^{1,\infty}$ and because of Theorem 5.1 we know that $W^*_{\mathcal{L}^\alpha}$ is bounded in $BMO_\sigma(\upsilon)$.

Now, if we define the transformation $S(f)(x) = f(x)x^{\alpha/2}$, in view of (6.1), we have that $W^*_{\ell^\alpha} = S^{-1} \circ W^*_{\mathcal{L}^\alpha} \circ S$ and also, according to the above definition, $\upsilon = S(\omega)$.

Therefore, it suffices to show that S is an isomorphism of Banach spaces between $BMO_\sigma(\omega)$ and $BMO_\sigma(\upsilon)$.

We shall give the details only for the boundedness of S. Indeed, assume that $f \in BMO_\sigma(\omega)$ and let us prove that

$$\frac{1}{\upsilon(I)} \int_I |S(f)(x)| dx \lesssim \|f\|_{BMO_\sigma(\omega)}$$

for any σ-critical interval I, and that

$$\frac{1}{\upsilon(I)} \int_I |S(f)(x) - c_I| dx \lesssim \|f\|_{BMO_\sigma(\omega)},$$

holds for any subcritical interval I.

Since any $I = (a, b)$ critical or subcritical interval for σ is also local, we have that $x^{\alpha/2} \simeq a^{\alpha/2} \simeq b^{\alpha/2}$ for any $x \in I$, and hence $\int_I S(g) \simeq a^{\alpha/2} \int_I g$. In particular, $\upsilon(I) \simeq a^{\alpha/2}\omega(I)$. Clearly these observations imply the first of the above inequalities.

To prove the second, we write

$$\begin{aligned}\frac{1}{\upsilon(I)} \int_I |S(f)(x) - c_I| dx &\lesssim \frac{1}{a^{\alpha/2}\omega(I)} \int_I |f(y)y^{\alpha/2} - c_I y^{-\alpha/2} y^{\alpha/2}| dy \\ &\lesssim \frac{1}{\omega(I)} \int_I |f(y) - c_I y^{-\alpha/2}| dy \\ &\lesssim A + B,\end{aligned}$$

with

$$A \doteq \frac{1}{\omega(I)} \int_I |f(y) - c_I a^{-\alpha/2}| dy$$

and

$$B \doteq \frac{|c_I|}{\omega(I)} \int_I |a^{-\alpha/2} - y^{-\alpha/2}| dy.$$

Notice that choosing $c_I = a^{\alpha/2} f_I$, it easily follows that $A \lesssim \|f\|_{BMO_\sigma(\omega)}$.

As for B we observe that to estimate the integrand we can make use of the mean value theorem since the interval $I = (a, b)$ is such that $0 < a < b < 2a$ and hence

$$|a^{-\alpha/2} - y^{-\alpha/2}| \simeq |a - y| s^{-1-\alpha/2} \lesssim (b - a) a^{-1-\alpha/2}.$$

In this way we arrive to

$$B \lesssim \frac{1}{\omega(I)} a^{-1} |I|^2 |f_I|.$$

But, if we call $\tilde{I}$ to the interval $(a, 2a)$, we get

$$|I||f_I| \leq \int_{\tilde{I}} |f| \leq \|f\|_{BMO_\sigma(\omega)} \omega(\tilde{I}).$$

Finally, as it is clear from inequality (6.3), ω is in A^1_{loc} and then is doubling over local intervals, so we have

$$\omega(\tilde{I}) \leq C\omega(I) \frac{|\tilde{I}|}{|I|},$$

leading to the desired inequality for B. $\square$

As it is easy to check either directly from (6.3) or through Corollary 5.2, the range for power weights x^δ is in this case $-\alpha - 1 < \delta \leq 0$, that is, the same power weights that belong to $A_1(x^\alpha dx)$.

References

[1] J. Betancor, J.C. Farina, L. Rodriguez-Mesa, A. Sanabria and J.L. Torrea, *Transference between Laguerre and Hermite settings*, J. Funtional Analysis **254** (2008), 826–850.

[2] Bennett, Colin and DeVore, Ronald A. and Sharpley, Robert, *Weak-L^∞ and BMO*, Annals of Mathematics. Second Series, **113** (1981), no. 3, 601–611.

[3] Anibal Chicco Ruiz and Eleonor Harboure, *Weighted norm inequalities for heat-diffusion Laguerre's semigroups*, Mathematische Zeitschrift, **257** (2007), no. 2, 329–354.

[4] Anibal Chicco Ruiz and Eleonor Harboure, *Weighted local BMO spaces and the local Hardy-Littlewood maximal operator*, Revista de la Unión Matemática Argentina. **52** (2011), no. 1, 47–56.

[5] L.Cha and H.P.Liu, *BMO spaces for Laguerre expansions*, Taiwanese Journal of Mathematics, **15** (2012), no. 6, 2153–2186.

[6] L.Cha and H.P.Liu, *BMO Boundedness of Maximal operators and g-Functions associated with Laguerre Expansions*, Journal of Funtion Spaces and Applications, **doi**:10.115/2012/923874.

[7] Dziubański, Jacek, *Hardy spaces for Laguerre expansions*, Constructive Approximation, **527** (2008), no. 3, 269–287.

[8] Dziubański, J. and Garrigós, G. and Martínez, T. and Torrea, J. L. and Zienkiewicz, J., *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Mathematische Zeitschrift, **249** (2005), no. 2, 329–356.

[9] Erdélyi, Arthur and Magnus, Wilhelm and Oberhettinger, Fritz and Tricomi, Francesco G., *Higher transcendental functions. Vols. I, II*, McGraw-Hill Book Company, Inc., New York-Toronto-London, (1953).

[10] Macías, R. and Segovia, C. and Torrea, J. L., *Heat-diffusion maximal operators for Laguerre semigroups with negative parameters*, Journal of Functional Analysis, **229** (2005), no. 2, 300–316.

[11] Macías, R. and Segovia, C. and Torrea, J. L., *Weighted norm estimates for the maximal operator of the Laguerre functions heat diffusion semigroup*, Studia Mathematica, **172** (2006), no. 2, 149–167.

[12] Nowak, Adam, *Heat-diffusion and Poisson integrals for Laguerre and special Hermite expansions on weighted L^p spaces*, Studia Mathematica, **158** (2003), no. 3, 239–268.

[13] Adam Nowak and Krzysztof Stempak, *Weighted estimates for the Hankel transform transplantation operator*, Tohoku Math. J. (2) **58** (2006), no. 2, 277–301.

[14] Stempak, Krzysztof, *Heat-diffusion and Poisson integrals for Laguerre expansions*, The Tohoku Mathematical Journal. Second Series, **46** (1994), no. 1, 83–104.

[15] Watson, G. N., *A treatise on the Theory of Bessel Functions*, Cambridge University Press (1966).

Aníbal Chicco Ruiz
Eleonor Harboure
Instituto de Matemática Aplicada del Litoral
CONICET-Universidad Nacional del Litoral
Santa Fe, Argentina.
anibalchicco@gmail.com
harbour@santafe-conicet.gov.ar